PROBLEMS

of Phystech.International 2018 Olympiad of Physics and Mathematics

Study Aid

Moscow
MIPT
2019

Moscow Institute of Physics and Technology

UDC 53
BBK 22.3
P61

P61 **Problems of Phystech.International—2018. Olympiad of Physics and Mathematics** : study aid / Chivilev V., Ouskov V., Sheronov A., Yuriev Yu., Plis V., Agakhanov N., Glukhov I., Gorodetskiy S., Podlipskii O. – M.: MIPT, 2019. – 51 pp.

This booklet presents problems that were offered at the final stage of Phystech.International Olympiad held in December 2018. (2018–2019 academic year).

All problems are provided with answers, some with detailed solutions. A problem-solving session lasted 4.5 hours.

These problems are intended for applicants to MIPT or other technical universities, as well as for teachers of schools with advanced study of physics and mathematics.

UDC 53
BBK 22.3

BBK means "Russian library and bibliographic classification" (BBK) index

Contents

PHYSICS

Problem Set #1, 9th Grade

1. Before an intersection, a lorry slows down to full stop in $T = 5$ s, in the middle of the braking distance its velocity equals $V = 2$ m/s.

1) Determine the lorry acceleration a during the slowdown.
2) Determine the lorry braking distance S.

During the slowdown the lorry goes in a straight line and its acceleration remains constant.

2. A tennis player is practicing on a horizontal court by sending the ball to a vertical wall. After a strike, the ball starts from the ground and hits the wall after $\tau_1 = 1.5$ s. In $\tau_2 = 0.5$ after an elastic collision with the wall, the ball lands on the court at a distance $S = 8$ m from the wall.

1) What is the maximum height H reached by the ball during the flight?
2) At what distance L from the wall has the ball started its flight?
3) Determine a velocity V_0 of the racket before the strike. The racket mass is much greater than the ball mass.

The collision of the racket and the ball is elastic. The ball is at rest before the strike. The gravitational acceleration is $g = 10$ m/s^2. The ball moves in a vertical plane perpendicular to the wall.

3. A ball made of cork is attached by a thread to the bottom of a vessel filled with water. The ball volume is V. The specific weight of water is ρ, the specific weight of cork is 0.2ρ. The gravitational acceleration is g.

1) Determine the thread tension T_1 if the vessel is at rest.
2) Determine the thread tension T_2 if the vessel moves on a horizontal surface at a constant acceleration $a = 0.5g$.

On both occasions, the ball is completely immersed in water and does not touch the walls.

4. Water vapor of mass $m_2 = 0.2$ kg and at $t_2 = 100$ °C is introduced in a vessel containing $m_1 = 10$ kg of water at $t_1 = 20$ °C. Determine the water temperature t_3 in the vessel after thermal equilibrium has been reached. The specific heat capacity of water is $c =$

$= 4200$ J/(kg·K), the specific heat of evaporation of water is $r = 2.26 \cdot 10^6$ J/kg. A heat capacity of the vessel and heat losses are negligible.

5. Six identical voltmeters are assembled in the circuit shown in the diagram, the circuit is connected to a power source of direct voltage of $U = 40$ V. The resistance of a voltmeter equals $R = 20 \cdot 10^3\ \Omega$.

1) Determine a current I_5 flowing through the voltmeter 5.

2) Determine the readings V_3 of the voltmeter 3.

3) Determine the readings V_1 of the voltmeter 1.

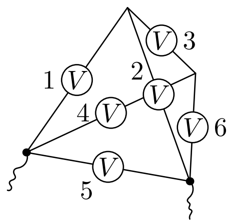

Fig. for problem 5

Problem Set #2, 9th Grade

1. A car moves on a horizontal road at $V_0 = 8$ m/s. Before an intersection, the car slows down to full stop in $T = 4$ s.

1) Determine a braking distance S of the car.

2) What was the car velocity V at the moment when it covered $\frac{15}{16}$ of the braking distance?

During the slowdown the car moves along a straight line and its acceleration remains constant.

2. A soccer player is practicing on a horizontal ground by sending the ball to a smooth vertical wall. After a strike, the ball starts from the ground and hits the wall in $\tau_1 = 0.4$ s. In $\tau_2 = 1.6$ s after an elastic recoil from the wall, the ball lands on the ground at a distance $S = 20$ m from the wall.

1) At which height H did the ball hit the wall?

2) At which distance L from the starting point did the ball land?

3) Determine an average force $\langle F \rangle$ exerted by the player on the ball during the strike.

The strike duration is $\Delta t = 0.05$ s, the ball mass is $m = 0.5$ kg. Before the strike the ball was at rest. The gravitational acceleration is $g = 10$ m/s^2. The ball moves in a vertical plane perpendicular to the wall.

3. A rubber air balloon is attached with a thread to the bottom of a steady vessel filled with water, the balloon is completely immersed in water. A thread tension is T_1. The specific weight of water is ρ, the gravitational acceleration is g.

1) Determine the volume V of the balloon. Neglect mass of the balloon with air inside.

If the vessel is being moved on horizontal surface along a straight line at a constant acceleration, the thread tension becomes equal T_2.

2) Determine the vessel acceleration a.

During the acceleration the ball is completely immersed in water and does not touch the walls.

4. A chunk of ice of mass $m = 1$ kg at $t_2 = -10$ °C is placed into a calorimeter containing $V = 5l$ of water at $t_1 = 90$ °C. Determine a temperature t in the calorimeter after thermal equilibrium is reached. The specific heat capacity of ice is $c_1 = 2100$ J/(kg·K), the specific heat capacity of water is $c_2 = 4200$ J/(kg·K), and the specific heat of fusion of ice is $\lambda = 3.3 \cdot 10^5$ J/kg. The specific water density is $\rho = 1.0 \cdot 10^3$ kg/m^3. The melting point of ice is at $t_0 = 0$ °C. Neglect a heat capacity of the calorimeter and a heat loss.

5. Six identical amperemeters are assembled in the circuit shown in the diagram and connected to a power source of direct voltage. Each amperemeter resistance is $R = 1\ \Omega$. A current displayed by the amperemeter 5 is $I = 5$ A.

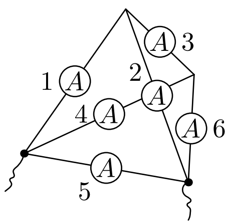

Fig. for problem 5

1) Determine a voltage U of the power source.

2) Determine a current I_3 displayed by the amperemeter 3.

3) Determine a current I_4 displayed by the amperemeter 4.

Problem Set #1, 10th Grade

1. A tennis player delivers a ball from the backline of a tennis court from a height of $H = 2m$, the ball velocity is horizontal. The court length is $L = 20$ m, the net height is $h = 1$ m. The ball flies over the net and lands on the tennis court.

1) Determine the duration T of the ball flight.
2) What is the minimum initial velocity V_0 at which the ball will fly over the net?
3) At which distance S from the net will the ball hit the court at this initial velocity?

The gravitational acceleration is $g = 10$ m/s^2. The ball moves in a vertical plane perpendicular to the net. Air resistance is negligible.

2. Masses of the weights 1, 2, and 3 in the diagram are $m_1 = 0.1m$ kg, $m_2 = 2m$, and $m_3 = 3m$.
1) Determine a tension T_1 of the thread on which the weight 3 is suspended.
2) Determine an acceleration a_1 of the weight 1.
3) Determine a tension T_2 of the thread on which the upper pulley is suspended.

The gravitational acceleration is $g = 10$ m/s^2. Weights 1 and 2 are attached to the threads. The pulleys are lightweight. Unstretcheable threads slide on the pulleys without friction.

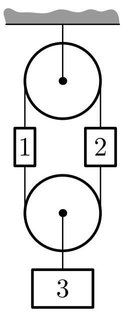

Fig. for problem 2

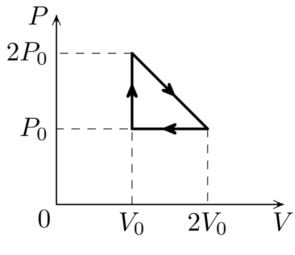

Fig. for problem 3

3. A monatomic gas is used as a working substance in the cycle shown in the Fig. Here, $P_0 = 1.0 \cdot 10^5$ Pa, $V_0 = 2.0 \cdot 10^{-3}$ m^3, the amount of the gas is $\nu = 2$ mol, and the molar mass of the gas is $\mu = 40 \cdot 10^{-3}$ kg/mol.
1) Determine a work A done by the gas per cycle.
2) Determine the minimum root mean squared (RMS) velocity of gas molecules $V_{\min}$ in the cycle.

3) By what percentage the maximum RMS exceeds the minimum RMS in this cycle?

The RMS velocity is defined as $V = \sqrt{\langle V^2 \rangle}$, where $\langle V^2 \rangle$ is the square of an atom velocity averaged over atoms of the gas.

4. Each of two small beads carries the same charge q, the beads are located on the inner surface of a smooth dielectric sphere of a radius R. The first bead is attached at the sphere lowest point and the second bead can freely slide on the surface. In equilibrium, the second bead resides at a height $h = 0.5 \cdot R$ measured from the lowest point.

1) Determine an electric force F acting between the charges.
2) Determine a mass m of the mobile bead.

The gravitational acceleration g and the electric constant ε_0 are known.

5. If a resistor $R = 50\ \Omega$ is connected to a battery, the current in the circuit equals I. If two resistors, R and an unknown $\tilde{R}$, are connected in series to the battery, the current in the circuit equals $0.75 \cdot I$. If the resistors R and $\tilde{R}$ are connected in parallel to the battery, the current through the battery equals $1.2 \cdot I$. Determine the resistor value of $\tilde{R}$.

Problem Set #2, 10^{th} Grade

1. A volleyball player delivers a ball from the volleyball court backline from a height H. The initial ball velocity is horizontal. The ball flies over the net and hits the court in $T = 0.8$ s. The court length is $L = 18$ m and the upper edge of the net is at a height of $h = 2.4$ m.

1) At what height H did the ball start?
2) What is the minimum initial velocity V_0 at which the ball will fly over the net?
3) At which distance S from the net will the ball hit the court at this velocity?

The gravitational acceleration is $g = 10\ \text{m/s}^2$. The ball moves in a vertical plane perpendicular to the net. Air resistance is negligible.

2. Masses of the weights 1, 2, and 3 in the diagram are $m_1 = m = 0.2$ kg, $m_2 = 2m$, and $m_3 = 5m$.

1) Determine a tension T of the thread on which the weight 3 is suspended.

2) Determine an acceleration a_2 of the weight 2.
3) What is the force P exerted by the upper pulley on the axis?

The gravitational acceleration is $g = 10$ m/s^2. Weights 1 and 2 are attached to the threads. The pulleys are lightweight, the axes are frictionless. The unstretcheable threads slide on the pulleys without friction.

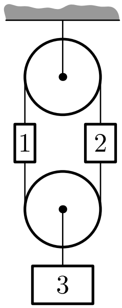

Fig. for problem 2

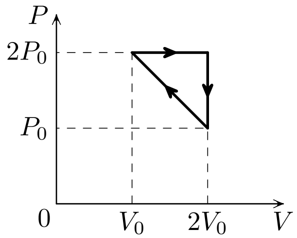

Fig. for problem 3

3. A monatomic ideal gas is used as a working substance in the cycle (see fig.), where $P_0 = 2.0 \cdot 10^5$ Pa, $V_0 = 1.0 \cdot 10^{-3}$ m^3, the gas amount is $\nu = 0.5$ mol, and its molar mass is $\mu = 20 \cdot 10^{-3}$ kg/mol.
1) Determine a work A done by the gas during the compression.
2) Determine the minimum root mean squared (RMS) velocity $V_{\min}$ of gas atoms in the cycle.

By what percentage does the maximum RMS exceeds the minimum RMS during the compression? The root mean squared velocity equals $V = \sqrt{\langle V^2 \rangle}$, where $\langle V^2 \rangle$—is the square of an atom velocity averaged over the atoms.

4. Each of two small beads carries a charge q, the beads are located on the inner surface of a smooth dielectric sphere of a radius R. The first bead is attached to the lowest point of the sphere and the second one can freely slide on the surface. In equilibrium, the distance between the beads equals R.
1) Determine the electric force F acting between the charges.
2) Determine the ratio $\eta = \dfrac{F}{mg}$ of the electric force to the weight of the mobile bead.

The gravitational acceleration g and the electric constant ε_0 are known.

5. If a resistor $R = 125\ \Omega$ is connected to a battery, the current I flows in the circuit. If the resistor R and unknown resistor $\tilde{R}$ are connected in series to the same battery, the current in the circuit is $\frac{3}{5}I$. If the resistors R and $\tilde{R}$ are connected in parallel to the battery, the current through the battery is $\frac{9}{8}I$. Determine a value of the resistor $\tilde{R}$.

Problem Set #1, 11th Grade

1. Blocks of masses $m_1 = m$ and $m_2 = 7m$ reside on a smooth horizontal surface. A light spring with a spring constant k is attached to the blocks, the spring is compressed by x (see the Fig.). The block of mass $7m$ is held at rest, the second block is pressed against the stop. Then the block of mass $7m$ is released.

Fig. for problem 1

1) Determine velocity of the block of mass 7m at the moment the second block is detaching from the stop.
2) After the blocks detached from the stop, determine a velocity of the block of mass $7m$ at a moment when the distance between the blocks is minimal.

2. A ball is suspended on an elastic spring and oscillates along the vertical with an amplitude A and a period T. The spring mass is negligible compared to the ball mass.

1) Determine magnitude of the maximum acceleration of the ball.
2) Determine magnitude of the ball acceleration when the magnitude of its velocity equals 2/3 of the magnitude of its maximum velocity.

3. One mole, $\nu = 1$, of helium expands first in the process 1–2 when the gas pressure P is directly proportional to its volume V and then in the process 2–3 specified by a linear dependence of the pressure on the volume (see Fig.). The pressure at the states 1 and 3 is equal. The work done by the gas in the process 2–3 equals 1.25 of the work done in the process 1–2. The temperature at the states 2 and 3 is the same and equal to $T_2 = 200$ K.

1) Determine a ratio of the volumes at the states 2 and 1.

2) Determine a work A_{23} done by the gas in the process 2–3.

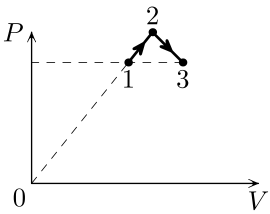

Fig. for problem 3

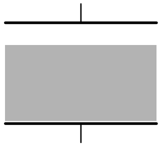
Fig. for problem 4

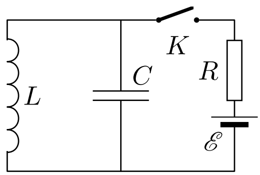

Fig. for problem 5

4. A parallel-plate capacitor is connected to a power source of a constant EMF. A rectangular slab of a dielectric material is inserted into the capacitor while the power source is connected to the capacitor (see the Fig.). The slab thickness equals 3/4 of the distance between the capacitor plates. As a result, the dielectric fills 3/4 of the capacitor volume and its capacitance increases three-fold.

1) How and by how much has the electric field inside the capacitor changed in the region free of dielectric?

2) Determine a permittivity of the dielectric slab.

5. In the circuit shown in the diagram, the switch K is initially opened. The circuit parameters are known, the inductor resistance is negligible compared to that of the resistor. Then the switch is closed. After a steady mode is achieved (the currents become constant), the switch is opened.

1) Determine a current I_1 through the capacitor right after the switch was closed.

2) Determine a current I_0 through the inductor in the steady mode when the switch is closed.

3) Determine a voltage across the capacitor after the switch was opened when the inductor current becomes equal to $\frac{2}{3} I_0$ at the first time.

Problem Set #2, 11th Grade

1. Two blocks of masses $m_1 = m$ and $m_2 = 5m$ reside on a smooth horizonal surface. A light elastic spring with a spring constant k is attached to the blocks, the spring is compressed by x (see the Fig.). The block of mass $5m$ is held at rest, the other block is pressed against the stop. Then the block of mass $5m$ is released.

Fig. for problem 1

1) Determine velocity of the block of mass $5m$ at the moment the other block is detaching from the stop.
2) After the blocks detached from the stop, determine a velocity of the block of mass $5m$ at a moment when the distance between the blocks is maximal.

2. A weight is suspended on an elastic spring and oscillates along the vertical with an amplitude A and a period T. A spring mass is negligible compared to a weight mass.
1) Determine the maximum magnitude of the weight acceleration.
2) Determine a magnitude of the weight acceleration at a moment, when the magnitude of its velocity equals 3/4 of the maximum.

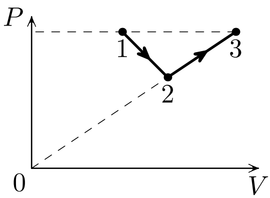

Fig. for problem 3

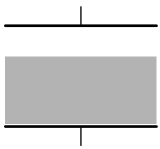
Fig. for problem 4

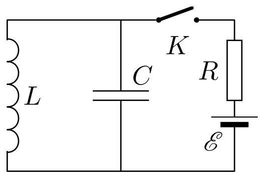

Fig. for problem 5

3. One mole, $\nu = 1$, of helium first expands in the process 1–2 when its pressure P is linearly related to its volume V, and then in the process 2–3 in which its pressure is directly proportional to the volume (see the Fig.). The pressure at the states 1 and 3 is the same. The work done by gas in the process 2–3 equals 1.5 of the work done in the process 1–2. The temperature at the states 1 and 2 is the same and equal to $T_1 = 200$ K.
1) Determine the ratio of gas volumes at the states 2 and 1.
2) Determine the work A_{23} done by the gas in the process 2–3.

4. A parallel-plate capacitor is connected to a power source of constant EMF. A rectangular dielectric slab is then inserted in the capacitor while the power source remains connected to the capacitor. The slab thickness is 2/3 of the distance between the capacitor plates. As a result, the dielectric fills 2/3 of the capacitor volume and its capacitance increases two-fold.

1) How and by how much has the electric field inside the capacitor changed in the region free of dielectric?
2) Determine a permittivity of the slab.

5. The switch K in the circuit shown in the diagram is initially opened. The values of the circuit elements are known, the inductor resistance is negligible compared to that of the resistor. Then the switch is closed. After a steady mode is achieved (the currents become constant), the switch is opened.

1) Determine current I_1 through the battery right after the switch was closed.
2) Determine current I_0 through the battery in the steady mode when the switch is closed.
3) After the switch was opened, determine a voltage across the capacitor at the moment when the current through the inductor becomes equal to $\frac{3}{4}\,I_0$ at the first time.

MATHEMATICS

Problem Set #1, 9th Grade

1. A passenger on a bus looks out at the window and sees his friend walking in the opposite direction. He gets off at the next stop, 3 minutes after having seen his friend, and starts walking in the opposite direction to catch up with him. How much time does he need for it (counting from the moment he got off the bus) if he walks 2.5 times faster than his friend but 6 times slower than the bus. (All speeds are constant.)

2. Jack and Jill exchanged some stamps from their collections two times. Each time Jill gave $\frac{2}{11}$ of all her stamps to Jack and Jack gave one half of his stamps to Jill (for example, if Jill had 11 stamps and Jack had 10 stamps, it means that at the first exchange Jill would give 2 stamps to Jack and Jack would give 5 stamps to Jill). It turned out that after the first exchange Jack had 110 stamps and after the second exchange Jill had 334 stamps. How many stamps did Jill have before all the exchanges?

3. An irreducible fraction, its numerator and denominator being positive integers, is greater than $\frac{1}{11}$. If its denominator is increased by 1 and its numerator is increased by 6 the resulting fraction is less than 0.2. Find the initial fraction if it is known that its denominator is 8 less than the square of its numerator.

4. Solve the system of equations $\begin{cases} xy^3 - x = 182, \\ xy^2 - xy = 42. \end{cases}$

5. The perimeter of a right triangle ABC is equal to 30. Hypotenuse AB touches the incircle of this triangle at point Q, and $AQ : QB = 10 : 3$. Find the area of this triangle.

6. Find all pairs of integers $(x; y)$ that satisfy inequalities

$$x^2 + 16y + 193 < 24x - y^2 \text{ and } 38x - y^2 > x^2 + 8y + 354.$$

7. KT is a bisector of triangle KLM. Circle Ω with its center on side KM has a radius of 6 and passes through points K, L and T. Find KL if it is known that $LT : MT = 1 : 3$.

Problem Set #2, 9th Grade

1. A passenger on a bus looks out at the window and sees his friend walking in the opposite direction. He gets off at the next stop, 1.5 minutes after having seen his friend, and starts walking in the opposite direction to catch up with him. How much time does he need for it (counting from the moment he got off the bus) if he walks 1.8 times faster than his friend but 11 times slower than the bus. (All speeds are constant.)

2. Jack and Jill exchanged some stamps from their collections two times. Each time Jill gave $\frac{3}{7}$ of all her stamps to Jack and Jack gave one third of his stamps to Jill (for example, if Jill had 14 stamps and Jack had 12 stamps, it means that at the first exchange Jill would give 6 stamps to Jack and Jack would give 4 stamps to Jill). It turned out that after the first exchange Jack had 273 stamps and after the second exchange Jill had 199 stamps. How many stamps did Jill have before all the exchanges?

3. An irreducible fraction, its numerator and denominator being positive integers, is greater than $\frac{1}{9}$. If its numerator is increased by 3 and its denominator is increased by 1 the resulting fraction is less than 0.2. Find the initial fraction if it is known that its denominator is 3 less than the square of its numerator.

4. Solve the system of equations $\begin{cases} x + xy^3 = -70, \\ xy + xy^2 = 20. \end{cases}$

5. The perimeter of a right triangle ABC is equal to 40. Hypotenuse AB touches the incircle of this triangle at point Q, and $AQ : QB = 5 : 12$. Find the area of this triangle.

6. Find all pairs of integers $(x; y)$ that satisfy inequalities

$$x^2 + 26y + 159 < 4x - y^2 \text{ and } 18x - y^2 > x^2 + 18y + 140.$$

7. KT is a bisector of triangle KLM. Circle Ω with its center on side KM has a radius of 6 and passes through points K, L and T. Find KL if it is known that $LT : MT = 2 : 3$.

Problem Set #1, 11th Grade

1. Solve the equation $x^4 \cdot 2^{11-x} + 2^{2+\sqrt{2x+2}} = x^4 \cdot 2^{\sqrt{2x+2}} + 2^{13-x}$.

2. Numbers a, b and c (in the indicated order) form a geometric progression. Numbers $c-a$, $2a-b$ and $a+b$ (in the indicated order) form an arithmetic progression. Find the common ratio of the geometric progression.

3. Find the value of $\cos 10^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ$.

4. Find the coordinates of point M such that it lies on y-axis and tangent lines drawn from point M to parabola $y = 7 - 5x - 3x^2$ are perpendicular to each other.

5. Point P belongs to side LM of parallelogram $KLMN$. It is known that $LP = MP = 2$, $\angle KPN = \arccos \frac{11}{12}$, $KL = 9$. Find the area of this parallelogram.

6. Sketch the set of points whose coordinates $(x;y)$ satisfy the system of inequalities

$$\begin{cases} \log_{|y-2|-2|y-4|+6}(x+3) > \log_{|y-2|-2|y-4|+6}(1+y), \\ x < 13. \end{cases}$$

Find the area of this set.

7. Isosceles trapezoids $APRS$ and $PQRS$, their largest bases being PR and PS respectively ($PR = PS$), are inscribed into circle Ω. Diagonals of trapezoid $PQRS$ intersect at point O, and angle POS equals 120°. Find the radius of Ω given that the area of triangle APS is equal to $4 + 4\sqrt{3}$.

Problem Set #2, 11th Grade

1. Solve the equation $x^4 \cdot 3^{\sqrt{1-3x}} + 3^{x+11} = x^4 \cdot 3^{x+9} + 3^{2+\sqrt{1-3x}}$.

2. Numbers a, b and c (in the indicated order) form a geometric progression. Numbers $3c-2a$, $a-b$ and $a-2c$ (in the indicated order) form an arithmetic progression. Find the common ratio of the geometric progression.

3. Find the value of $\sin 10^\circ \cdot \sin 50^\circ \cdot \sin 70^\circ$.

4. Find the coordinates of point M such that it lies on y-axis and tangent lines drawn from point M to parabola $y = 1 + 6x - 4x^2$ are perpendicular to each other.

5. Point P belongs to side LM of parallelogram $KLMN$. It is known that $LP = MP = 3$, $\angle KPN = \arccos\frac{8}{9}$, $KL = 5$. Find the area of this parallelogram.

6. Sketch the set of points whose coordinates $(x; y)$ satisfy the system of inequalities

$$\begin{cases} \log_{|x+2|-2|x+3|+4}(y-2) > \log_{|x+2|-2|x+3|+4}(1-x), \\ y < 13. \end{cases}$$

Find the area of this set.

7. Isosceles trapezoids $APRS$ and $PQRS$, their largest bases being PR and PS respectively ($PR = PS$), are inscribed into circle Ω. Diagonals of trapezoid $APRS$ intersect at point O, and angle AOP equals $60°$. Find the radius of Ω given that the area of triangle PQR is equal to $9 + 9\sqrt{3}$.

PHYSICS

Assessment Criteria

For the Final Stage of *Phystech.International* Olympiad

December, 2018.

Maximal total points for each problem: 10.

Problem Sets 1 and 2 for Pre-Graduation Class (9-th Grade)

Problem	Assessment Criteria	Num pts
1.	1) Correct answer to the first question	5
	2) Correct answer to the second question	5
2.	1) Correct answer to the first question	3
	2) Correct answer to the second question	3
	3) Correct answer to the third question	4
3.	1) Correct answer to the first question	4
	2) Correct answer to the second question	6
4.	Heat balance equation is correctly written	7
	Correct answer is presented	3
5.	An equivalent scheme is presented or described in words	4
	1) Correct answer to the first question	2
	2) Correct answer to the second question	2
	3) Correct answer to the third question	2

Problem Sets 1 and 2 for Pre-Graduation Class (10-th Grade)

Problem	Assessment Criteria	Num pts
1.	1) Correct answer to the first question	2
	2) Correct answer to the second question	4
	3) Correct answer to the third question	4
2.	1) Correct answer to the first question	2
	2) Correct answer to the second question	4
	3) Correct answer to the third question	4

3.	1) Correct answer to the first question	2
	2) Correct answer to the second question	2 (PS #1)
		6 (PS #2)
	3) Correct answer to the third question	6 (PS #1)
		2 (PS #2)
4.	1) Correct answer to the first question	5
	2) Correct answer to the second question	5
5.	All the equations are written correctly	7
	Correct answer is presented	3

Problem Sets 1 and 2 for Graduation Class (11-th Grade)

Problem	Assessment Criteria	Num pts
1.	1) Correct answer to the first question	5
	2) Correct answer to the second question	5
2.	1) Correct answer to the first question	3
	2) Correct answer to the second question	7
3.	All the required equations are written correctly	4
	1) Correct answer to the first question	3
	2) Correct answer to the second question	3
4.	1) Correct answer to the first question	5
	2) Correct answer to the second question	5
5.	1) Correct answer to the first question	3
	2) Correct answer to the second question	3
	3) Correct answer to the third question	4

PHYSICS

Problem Set #1, 9th Grade

1. $$\left.\begin{aligned} S &= \frac{aT^2}{2} \\ \frac{S}{2} = \frac{V^2}{2a};\ S &= \frac{V^2}{a} \end{aligned}\right\} \Rightarrow a = \sqrt{2}\,\frac{V}{T} = \sqrt{2}\,\frac{2}{5} \approx 0.57\ \text{m/s}^2.$$

$$S = \frac{VT}{\sqrt{2}} = \frac{2\cdot 5}{\sqrt{2}} \approx 7.1\ \text{m}.$$

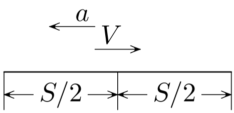

Fig. 1

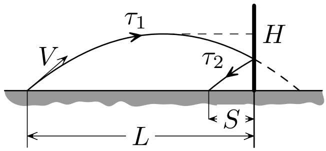

Fig. 2

2. $$H = \frac{g\left(\dfrac{\tau_1+\tau_2}{2}\right)^2}{2} = \frac{g(\tau_1+\tau_2)^2}{8} = \frac{10\cdot 4}{8} = 5\ \text{m}. \qquad (1)$$

$$V_x = \frac{S}{\tau_2}\,, \qquad (2)$$

$$L = V_x\tau_1\,. \qquad (3)$$

From (2) and (3),

$$L = \frac{S\tau_1}{\tau_2} = \frac{8\cdot 1.5}{0.5} = 24\ \text{m},$$

$$H = \frac{V_y^2}{2}\,\frac{1}{g}\,. \qquad (4)$$

From (1) and (4), $V_y = g\,\dfrac{\tau_1+\tau_2}{2}\,.$ (5)

Based on (2) and (5),

$$V = \sqrt{V_x^2+V_y^2} = \sqrt{\left(\frac{S}{\tau_2}\right)^2 + \left[\frac{g(\tau_1+\tau_2)}{2}\right]^2}, \quad V = V_0 + V_{\text{rel}} = 2V_0;$$

$$V_0 = \frac{1}{2}V = \frac{1}{2}\sqrt{\left(\frac{S}{\tau_2}\right)^2 + \left[\frac{g(\tau_1+\tau_2)}{2}\right]^2} =$$

$$= \frac{1}{2}\sqrt{\left(\frac{8}{0.5}\right)^2 + \left(\frac{10\cdot 2}{2}\right)^2} = \sqrt{89} \approx 9.4\ \text{m/s}.$$

3. 1) $F_A = g\rho V$. (1)

$F_A = T_1 + mg = T_1 + 0.2\rho V g$. (2)

From (1) and (2) $T_1 = 0.8 g\rho V$. (3)

2) $g' = \sqrt{g^2 + a^2}$ (see fig. 3b)).

Similarly to (3) we get

$$T_2 = 0.8 g'\rho V = 0.8\sqrt{1.25} g\rho V \approx 0.9 g\rho V.$$

4. From the heat balance equation

$$m_2 r + m_2 c(t_2 - t_3) = m_1 c(t_3 - t_1)$$

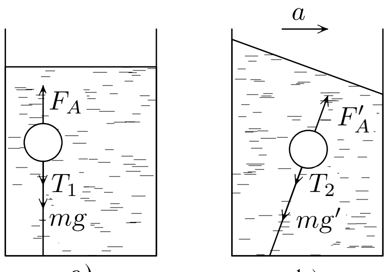

Fig. 3

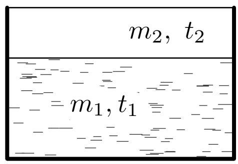

Fig. 4

we obtain

$$t_3 = \frac{m_1 t_1 + m_2 t_2}{m_1 + m_2} + \frac{m_2 r}{c(m_1 + m_2)} =$$

$$= \frac{10 \cdot 20 + 0.2 \cdot 100}{10 + 0.2} + \frac{0.2 \cdot 2.26 \cdot 10^6}{4200(10 + 0.2)} \approx 32\ ^\circ\text{C}.$$

5. An equivalent scheme is shown in the fig. 5.

1) $I_5 = \dfrac{U}{R} = \dfrac{40}{20{\cdot}10^3} = 2\cdot 10^{-3}$ A $= 2$ mA.

2) From the symmetry $V_3 = 0$.

3) $V_1 = \dfrac{1}{2} U = \dfrac{40}{2} = 20$ B.

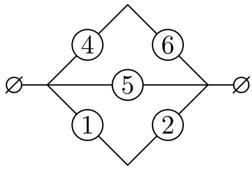

Fig. 5

Problem Set #2, 9th Grade

1. $S = \dfrac{aT^2}{2} = \dfrac{aT \cdot T}{2} = \dfrac{V_0 T}{2} = \dfrac{8 \cdot 4}{2} = 16$ m (see fig. 6) (1)

$$a = \frac{V_0}{T}\,,\ \frac{S}{16} = \frac{V^2}{2a} = \frac{V^2}{2(V_0/T)}\,.$$

With (1) taken into account, $V = \sqrt{\dfrac{SV_0}{8T}} = \dfrac{V_0}{4} = 2$ m/s.

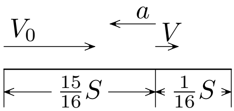

Fig. 6

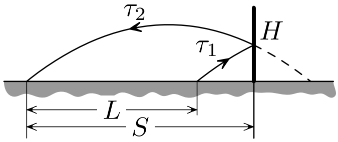

Fig. 7

2. $$H = \frac{g\left(\frac{\tau_1+\tau_2}{2}\right)^2}{2} = \frac{g(\tau_1+\tau_2)^2}{8} = \frac{10\cdot 4}{8} = 5\ \text{m}. \quad (1)$$

$$V_x = \frac{S}{\tau_2}\,, \quad (2)$$

$$S - L = V_x\tau_1 \quad \text{(see fig. 7)} \quad (3)$$

From (2) and (3)

$$L = \left(1-\frac{\tau_1}{\tau_2}\right)S = \left(1-\frac{0.4}{1.6}\right)S = \frac{3}{4}\,S = 15\ \text{m},$$

$$H = \frac{V_y^2}{2}\,\frac{1}{g}\,. \quad (4)$$

Based on (1) and (4), $V_y = \dfrac{g(\tau_1+\tau_2)}{2}$. (5)

From (2) and (5), $V = \sqrt{V_x^2+V_y^2} = \sqrt{\left(\dfrac{S}{\tau_2}\right)^2 + \left[\dfrac{g(\tau_1+\tau_2)}{2}\right]^2}$.

$$\langle F\rangle\Delta t = mV;$$

$$\langle F\rangle = \frac{mV}{\Delta t} = \frac{m}{\Delta t}\sqrt{\left(\frac{S}{\tau_2}\right)^2 + \left[\frac{g(\tau_1+\tau_2)}{2}\right]^2} =$$

$$= \frac{0.5}{0.05}\sqrt{\left(\frac{20}{1.6}\right)^2 + \left(\frac{10\cdot 2}{2}\right)^2} \approx 160\ \text{N}.$$

3. 1) $F_A = g\rho V$, (1)

$F_A = T_1$. (2)

From (1) and (2) $T_1 = g\rho V$; $V = \dfrac{T_1}{g\rho}$. (3)

2) $g' = \sqrt{g^2+a^2}$.

Similarly to (3) we obtain

$$T_2 = g'\rho V = \sqrt{g^2+a^2}\,\rho V \Rightarrow a = \sqrt{\left(\frac{T_2}{\rho V}\right)^2 - g^2}.$$

4. From the heat balance equation

$$mc_1(t_0-t_2) + \lambda m + mc_2(t-t_0) = \rho V c_2(t_1-t)$$

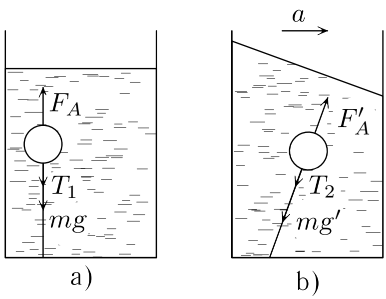

Fig. 8

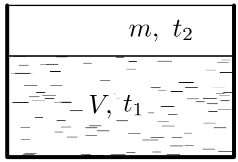

Fig. 9

we obtain

$$t = \frac{c_2\rho V t_1 + mc_1t_2 - \lambda m}{c_2(\rho V + m)} =$$

$$= \frac{4200 \cdot 10^3 \cdot 5 \cdot 10^{-3} \cdot 90 + 1 \cdot 2100 \cdot (-10) - 3.3 \cdot 10^5 \cdot 1}{4200 \cdot (10^3 \cdot 5 \cdot 10^{-3} + 1)} \approx 61\ ^\circ\text{C}.$$

5. An equivalent scheme is shown in the fig. 10.

1) $U = IR = 5B$.

2) From the symmetry, $I_3 = 0$.

3) $I_4 = \dfrac{U}{2R} = \dfrac{IR}{2R} = \dfrac{I}{2} = 2.5$ A.

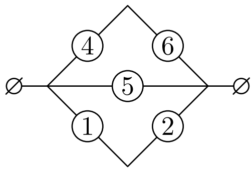

Fig. 10

Problem Set #1, 10th Grade

1. 1) $H = \dfrac{gT^2}{2} \Rightarrow T = \sqrt{\dfrac{2H}{g}} =$

$= \sqrt{\dfrac{2 \cdot 2}{10}} \approx 0.63$ c. (1)

2) Let t be a time of ball flight to the net. Then

$\dfrac{L}{2} = V_0 t\,; \quad t = \dfrac{L}{2V_0}\,.$ (2)

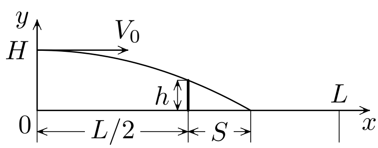

Fig. 11

$$h = H - \frac{gt^2}{2} = H - \frac{g}{2} \cdot \frac{L^2}{4V_0^2} = H - \frac{gL^2}{8V_0^2} \Rightarrow$$

$$\Rightarrow V_0 = L\sqrt{\frac{g}{8(H-h)}} = 20\sqrt{\frac{10}{8 \cdot (2-1)}} \approx 22.4\ \text{m/s}. \qquad (3)$$

3) Based on (1) and (3) we get

$$S = V_0T - \frac{L}{2} = L\sqrt{\frac{g}{8(H-h)} \cdot \frac{2H}{g}} - \frac{L}{2};$$

$$S = \frac{L}{2}\left(\sqrt{\frac{H}{H-h}} - 1\right) = 10 \cdot (\sqrt{2} - 1) \approx 4.1 \text{ m}.$$

2. 1) $T_1 = m_3g = 3mg = 3 \cdot 0.1 \cdot 10 = 3\ \text{H}.$ (1)

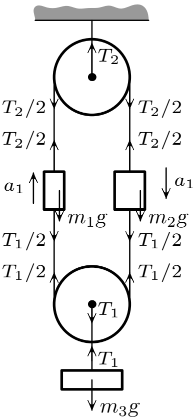

Fig. 12

2) $m_1a_1 = \frac{1}{2}(T_2 - T_1) - m_1g.$ (2)

$m_2a_1 = \frac{1}{2}(T_1 - T_2) + m_2g.$ (3)

From (2) and (3)

$a_1 = \frac{m_2 - m_1}{m_1 + m_2}\, g = \frac{g}{3} \approx 3.3\ \text{m/s}^2.$ (4)

3) (1), (4) → (2): $T_2 = \left(\frac{4m_1m_2}{m_1+m_2} + m_3\right) g = \left(\frac{4 \cdot 0.1 \cdot 0.2}{0.3} + 0.3\right) \cdot 10 \approx 5.7\ \text{H}.$

3. 1) $A = \frac{1}{2} P_0V_0 = \frac{10^5 \cdot 2 \cdot 10^{-3}}{2} = 100\ \text{J}.$

2) $P_0V_0 = \nu RT_{\min}$; $T_{\min} = \frac{P_0V_0}{\nu R}.$ (1)

Taking into account (1),

$$v_{\min} = \sqrt{\frac{3RT_{\min}}{\mu}} = \sqrt{\frac{3RP_0V_0}{\mu\nu R}} = \sqrt{\frac{3P_0V_0}{\mu\nu}} = \sqrt{\frac{3 \cdot 10^5 \cdot 2 \cdot 10^{-3}}{2 \cdot 40 \cdot 10^{-3}}} \approx 87 \text{ m/s}.$$

3) In section 1–2, $P = 2P_0 - \frac{P_0}{V_0}(V - V_0) = -\frac{P_0}{V_0}V + 3P_0,$

$\nu RT = PV = -\frac{P_0}{V_0}V^2 + 3P_0V.$

$T_{\max}$ is achieved at $V = \frac{3}{2}V_0,$

$P = \frac{3}{2}P_0:\ \nu RT_{\max} = -\frac{P_0}{V_0} \cdot \frac{9}{4}V_0^2 + 3P_0\frac{3}{2}V_0 = \frac{9}{4}P_0V_0;$

$T_{\max} = \frac{9P_0V_0}{4\nu R}.$ (2)

With (2) taken into account

$$v_{\max} = \sqrt{\frac{3RT_{\max}}{\mu}} = \sqrt{\frac{3R}{\mu} \cdot \frac{9P_0V_0}{4\nu R}} = \frac{3}{2}\sqrt{\frac{3P_0V_0}{\mu\nu}} = \frac{3}{2}v_{\min}$$

(an increase by 50%).

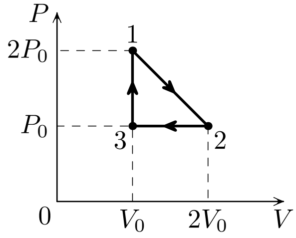

Fig. 13

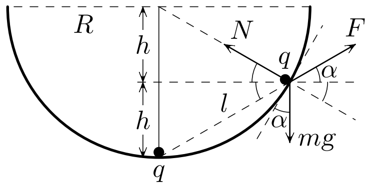

Fig. 14

4. It follows from the equality of rectangular triangles that $l = R$. $\sin\alpha = \dfrac{h}{R} = 0.5$; $\alpha = 30°$ (see fig. 14).

1) $F = \dfrac{q^2}{4\pi\varepsilon_0 l^2} = \dfrac{q^2}{4\pi\varepsilon_0 R^2}$.

2) $mg\cos\alpha = F\sin 2\alpha$, $m = \dfrac{F\cdot 2\sin\alpha\cdot\cos\alpha}{g\cos\alpha} = \dfrac{2F\sin\alpha}{g} =$
$= \dfrac{2F\cdot 0.5}{g} = \dfrac{F}{g}$.

5. Let r be a battery internal resistance. Then

$$\frac{\mathscr{E}}{I} = r + R, \tag{1}$$

$$\frac{\mathscr{E}}{I} = 0.75(r + R + \tilde{R}), \tag{2}$$

$$\frac{\mathscr{E}}{I} = 1.2\left(r + \frac{R\tilde{R}}{R+\tilde{R}}\right). \tag{3}$$

From (1) and (2), $r = 3\tilde{R} - R$. (4)

From (1) and (3), $R = 0.2r + 1.2\,\dfrac{R\tilde{R}}{R+\tilde{R}}$. (5)

Based on (4) and (5), $\tilde{R}^2 + R\tilde{R} - 2R^2 = 0$,

$$\tilde{R} = \frac{-R + \sqrt{R^2 + 8R^2}}{2} = \frac{-R + 3R}{2} = R = 50 \text{ Ohm}.$$

Problem Set #2, 10th Grade

1. 1) $H = \frac{gT^2}{2} = \frac{10 \cdot 0.8^2}{2} = 3.2$ m. (1)

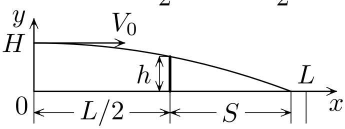

Fig. 15

2) Let t be a time of ball flight to the net. Then

$$\frac{L}{2} = V_0 t\,; \qquad t = \frac{L}{2V_0}\,. \qquad (2)$$

$$h = H - \frac{gt^2}{2} = \frac{gT^2}{2} - \frac{g}{2}\cdot\frac{L^2}{4V_0^2} \Rightarrow$$

$$\Rightarrow V_0 = \frac{L}{2\sqrt{T^2 - \frac{2h}{g}}} = \frac{18}{2\sqrt{0.8^2 - \frac{2\cdot 2.4}{10}}} = 22.5 \text{ m/s}. \qquad (3)$$

3) With (3) taken into account,

$$S = V_0 T - \frac{L}{2} = \frac{LT}{2\sqrt{T^2 - \frac{2h}{g}}} - \frac{L}{2} =$$

$$= \frac{L}{2}\left(\frac{1}{\sqrt{1 - \frac{2h}{gT^2}}} - 1\right) = \frac{18}{2}\left(\frac{1}{\sqrt{1 - \frac{2\cdot 2.4}{10\cdot 0.8^2}}} - 1\right) = 9 \text{ m}.$$

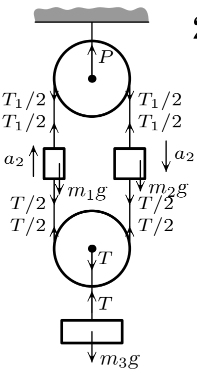

Fig. 16

2. 1) $T = m_3 g = 5mg = 5\cdot 0.2\cdot 10 = 10$ H. (1)

2) $m_1 a_2 = \frac{1}{2}(T_1 - T) - m_1 g\,,$ (2)

$m_2 a_2 = \frac{1}{2}(T - T_1) + m_2 g\,.$ (3)

From (2) and (3)

$a_2 = \frac{m_2 - m_1}{m_1 + m_2}\, g = \frac{g}{3} \approx 3.3$ m/s^2. (4)

3) (1), (4) → (2):

$$P = T_1 = \left(\frac{4m_1 m_2}{m_1 + m_2} + m_3\right) g =$$

$$= \left(\frac{4\cdot 0.2\cdot 0.4}{0.6} + 1\right)\cdot 10 \approx 15.3 \text{ H}.$$

3. 1) $A = A_{31} = -\frac{2P_0 + P_0}{2}\, V_0 = -\frac{3}{2}\, P_0 V_0 = -\frac{3}{2}\cdot 2\cdot 10^5\cdot 10^{-3} =$
$= -300$ J.

2) In section 3–1 (see fig. 17): $P = 2P_0 - \frac{P_0}{V_0}(V - V_0) = -\frac{P_0}{V_0}V + 3P_0\,.$

$T_{\min}$ is achieved at $V = \dfrac{3}{2}\,V_0$, $P = \dfrac{3}{2}\,P_0$:

$\nu R T_{\min} = -\dfrac{P_0}{V_0} \cdot \dfrac{9}{4}\,V_0^2 + 3P_0\,\dfrac{3}{2}\,V_0 = \dfrac{9}{4}\,P_0 V_0$;

$$T_{\min} = \frac{9}{4}\,\frac{P_0 V_0}{\nu R}\,. \qquad (1)$$

Reckoning (1) we obtain $v_{\min} = \sqrt{\dfrac{3RT_{\min}}{\mu}} = \sqrt{\dfrac{3R}{\mu}\,\dfrac{9}{4}\,\dfrac{P_0V_0}{\nu R}} =$
$= \dfrac{3}{2}\sqrt{\dfrac{3P_0V_0}{\nu \cdot \mu}} = \dfrac{3}{2}\sqrt{\dfrac{3 \cdot 2 \cdot 10^5 \cdot 10^{-3}}{0.5 \cdot 20 \cdot 10^{-3}}} \approx 367$ m/s.

3) $T_{\max}$ is achieved at $V = 2V_0$, $P = 2P_0$: $\nu R T_{\max} = 4P_0V_0$;

$T_{\max} = \dfrac{4P_0V_0}{\nu R}$,

$$v_{\max} = \sqrt{\frac{3RT_{\max}}{\mu}} = \sqrt{\frac{3R}{\mu}\,\frac{4P_0V_0}{\nu R}} = 2\sqrt{\frac{3P_0V_0}{\mu\nu}},$$

$\dfrac{v_{\max}}{v_{\min}} = \dfrac{2}{3/2} = \dfrac{4}{3} \approx 1.33$ (an increase by $\approx 33\,\%$).

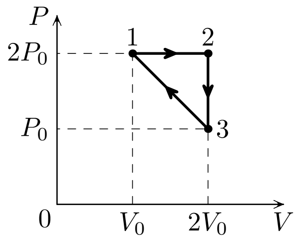

Fig. 17

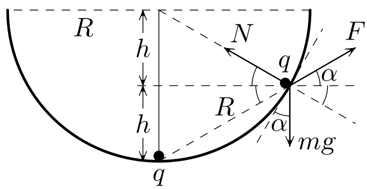

Fig. 18

4. Since the triangle in the fig. 18 is equilateral then

$$\alpha = \frac{1}{2}\,60^\circ = 30^\circ\,. \qquad (1)$$

1) $F = \dfrac{q^2}{4\pi\varepsilon_0 R^2}$.

2) $mg\cos\alpha = F\sin 2\alpha \Rightarrow \eta = \dfrac{F}{mg} = \dfrac{\cos\alpha}{\sin 2\alpha} = \dfrac{1}{2\sin\alpha} =$
$= \dfrac{1}{2 \cdot 0.5} = 1$.

5. Let r be a battery internal resistance. Then

$$\frac{\mathscr{E}}{I} = r + R\,, \qquad (1)$$

$$\frac{\mathscr{E}}{I} = \frac{3}{5}\,(r + R + \tilde{R})\,, \tag{2}$$

$$\frac{\mathscr{E}}{I} = \frac{9}{8}\left(r + \frac{R\tilde{R}}{R + \tilde{R}}\right). \tag{3}$$

From (1) and (2), $r = \frac{3}{2}\,\tilde{R} - R\,.$ (4)

From (1) and (3), $R = 8R - 9\,\frac{R\tilde{R}}{R + \tilde{R}}\,.$ (5)

Based on (4) and (5), $\tilde{R}^2 + R\tilde{R} - 6R^2 = 0$, and

$$\tilde{R} = \frac{-R + \sqrt{R^2 + 24R^2}}{2} = 2R = 2 \cdot 125 = 250 \text{ Ohm}.$$

Problem Set #1, 11th Grade

1. 1) When block m_1 is detached from the stop the spring is not compressed. From energy conservation

$$\frac{kx^2}{2} = \frac{m_2 v_2^2}{2} \Rightarrow v_2 = x\sqrt{\frac{k}{m_2}} = x\sqrt{\frac{k}{7m}}. \tag{1}$$

2) At the minimum distance between the blocks, their speed is the same and equal to v (see fig. 19).
From momentum conservation, $m_2 v_2 = (m_1 + m_2)v$;
With (1) taken into account,

$$v = \frac{m_2}{m_1 + m_2} v_2 = \frac{7}{8} x\sqrt{\frac{k}{7m}} = \frac{x}{8}\sqrt{\frac{7k}{m}}.$$

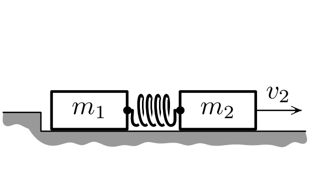

Fig. 19

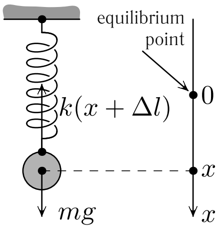

Fig. 20

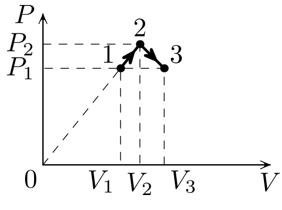

Fig. 21

2. 1) $mg = k\Delta l$ where Δl is the deformation in equilibrium.

$$m\ddot{x} = -k(x + \Delta l) + mg = -kx,$$

$$\ddot{x} + \omega_0^2 x = 0, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}, \quad (1)$$

$$x = A\cos(\omega_0 t); \quad \ddot{x} = -\omega_0^2 A\cos(\omega_0 t), \quad (2)$$

$$|a_{\max}| = \omega_0^2 A = \frac{4\pi^2}{T^2}A.$$

2) $\dot{x} = -\omega_0 A\sin(\omega_0 t)$,

$$v = \frac{2}{3}v_{\max} \text{ if } \sin(\omega_0 t) = \frac{2}{3} \Rightarrow \cos(\omega_0 t) = \sqrt{1 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}. \quad (3)$$

(3)→(2): $\ddot{x} = -\omega_0^2 A\,\frac{\sqrt{5}}{3}$; $|a| = \frac{\sqrt{5}}{3}\,\omega_0^2 A = \frac{\sqrt{5}}{3}\,\frac{4\pi^2}{T^2}\,A$.

3. Let $\frac{P_2}{P_1} = \frac{V_2}{V_1} = \alpha$. Then

$$A_{12} = \frac{1}{2}(P_1 + P_2)(V_2 - V_1) =$$

$$= \frac{1}{2}(P_1V_2 - P_1V_1 + P_2V_2 - P_2V_1) = \frac{R}{2}(T_2 - T_1), \quad (1)$$

$$A_{23} = \frac{1}{2}(P_1 + P_2)(V_3 - V_2) = \frac{1}{2}(P_1V_3 - P_1V_2 + P_2V_3 - P_2V_2) =$$

$$= \frac{1}{2}(\alpha RT_3 - \alpha RT_1) = \frac{\alpha R}{2}(T_2 - T_1). \quad (2)$$

From (1) and (2), $\frac{A_{23}}{A_{12}} = \frac{5}{4} = \alpha$. (3)

$\frac{V_2}{V_1} = \alpha = \frac{5}{4}$, $RT_1 = P_1V_1 = \frac{P_2}{\alpha}\cdot\frac{V_2}{\alpha} = \frac{1}{\alpha^2}RT_2$;

$T_1 = \frac{T_2}{\alpha^2} = \frac{16}{25}T_2$. (4)

(3), (4)→(2):

$$A_{23} = \frac{5R}{4\cdot 2}\left(T_2 - \frac{16}{25}T_2\right) = \frac{9}{40}RT_2 = \frac{9}{40}\cdot 8.31\cdot 200 = 374\text{ J}.$$

4. Initial capacity $C = \frac{\varepsilon_0 S}{d_0}$. (1)

In the 2nd case, a capacitor with a plate is equivalent to two capacitors connected in series (see fig. 22).

$$\frac{1}{C'} = \frac{1}{\varepsilon_0\,\frac{S}{d_0/4}} + \frac{1}{\varepsilon\varepsilon_0\,\frac{S}{3d_0/4}} \Rightarrow C' = \frac{4}{3+\varepsilon}\varepsilon\varepsilon_0\frac{S}{d_0}. \quad (2)$$

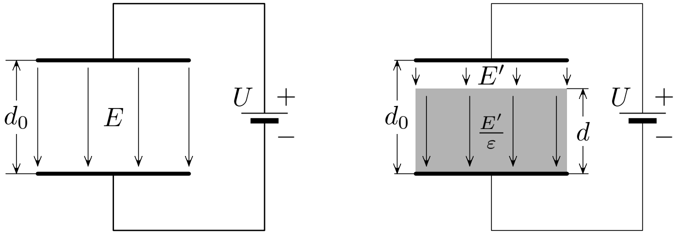

Fig. 22

From (1) and (2), $C'/C = \dfrac{4\varepsilon}{3+\varepsilon} = 3 \Rightarrow 4\varepsilon = 9 + 3\varepsilon\,;\ \varepsilon = 9\,.$ (3)

$$Ed_0 = E'\frac{d_0}{4} + \frac{E'}{\varepsilon}\cdot\frac{3}{4}d_0 = \frac{E'd_0}{4} + \frac{E'}{9}\cdot\frac{3}{4}d_0 \Rightarrow E' = 3E.$$

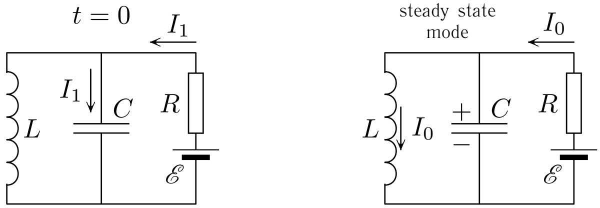

Fig. 23

5. 1) Immediately after closing the switch, the current through the coil is zero $\Rightarrow I_1 = \mathscr{E}/R$ (see Fig. 23a)).

2) In the steady state mode (see Fig. 23b)) $I_0 = \mathscr{E}/R\,.$ (1)

3) Energy conservation law: $\dfrac{LI_0^2}{2} = \dfrac{CU_C^2}{2} + \dfrac{L\left(\frac{2}{3}\,I_0\right)^2}{2}\,.$ (2)

(1) → (2): $L\,\dfrac{\mathscr{E}^2}{R^2} = CU_C^2 + L\,\dfrac{4}{9}\,\dfrac{\mathscr{E}^2}{R^2}\,,\ U_C = \dfrac{\sqrt{5}\mathscr{E}}{3\cdot R}\sqrt{\dfrac{L}{C}}\,.$

Problem Set #2, 11th Grade

1. 1) When block m_1 is detached from the stop the spring is not compressed. From energy conservation

$$\frac{kx^2}{2} = \frac{m_2v_2^2}{2} \Rightarrow v_2 = x\sqrt{\frac{k}{m_2}} = x\sqrt{\frac{k}{5m}}. \qquad (1)$$

2) At the maximum distance between the blocks, their speed is the same and equal to v (see fig. 24).
From momentum conservation, $m_2 v_2 = (m_1 + m_2)v$;

$$v = \frac{m_2}{m_1 + m_2} v_2 = \frac{5}{6} x \sqrt{\frac{k}{5m}} = \frac{x}{6}\sqrt{\frac{5k}{m}}.$$

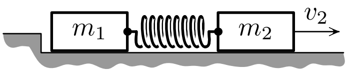

Fig. 24

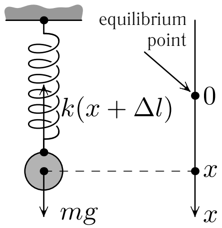

Fig. 25

2. 1) $mg = k\Delta l$ where Δl is the deformation in equilibrium (see fig. 25).

$$m\ddot{x} = -k(x + \Delta l) + mg = -kx,$$

$$\ddot{x} + \omega_0^2 x = 0, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}, \qquad (1)$$

$$x = A\cos(\omega_0 t); \quad \ddot{x} = -\omega_0^2 A\cos(\omega_0 t), \ (2) |a_{\max}| = \omega_0^2 A = \frac{4\pi^2}{T^2} A.$$

2) $\dot{x} = -\omega_0 A \sin(\omega_0 t)$,

$$v = \frac{3}{4} v_{\max} \text{ if } \sin(\omega_0 t) = \frac{3}{4} \Rightarrow \cos(\omega_0 t) = \sqrt{1 - \left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}. \quad (3)$$

$$(3) \to (2): \ \ddot{x} = -\omega_0^2 A \frac{\sqrt{7}}{4}; \ |a| = \frac{\sqrt{7}}{4}\omega_0^2 A = \frac{\sqrt{7}}{4}\frac{4\pi^2}{T^2} A.$$

3. Let $\frac{P_1}{P_2} = \frac{V_3}{V_2} = \alpha$ (see fig. 26). Then

$$RT_3 = P_1 V_3 = \alpha P_2 \cdot \alpha V_2 = \alpha^2 RT_1 \Rightarrow$$
$$\Rightarrow T_3 = \alpha^2 T_1. \qquad (1)$$

$$A_{12} = \frac{1}{2}(P_1 + P_2)(V_2 - V_1) =$$
$$= \frac{1}{2}(P_1 V_2 - P_1 V_1 + P_2 V_2 - P_2 V_1) =$$
$$= \frac{1}{2}\left(\alpha RT_2 - \frac{1}{\alpha} RT_1\right) = \frac{RT_1}{2}\frac{\alpha^2 - 1}{\alpha}. \qquad (2)$$

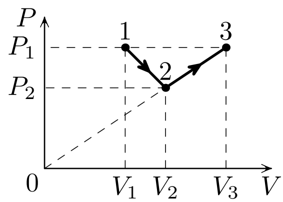

Fig. 26

$$A_{23} = \frac{1}{2}(P_1 + P_2)(V_3 - V_2) = \frac{1}{2}(P_1V_3 - P_1V_2 + P_2V_3 - P_2V_2) =$$
$$= \frac{1}{2}\left(RT_3 - \alpha RT_2 + \frac{1}{\alpha}RT_3 - RT_2\right) = \frac{R(\alpha+1)}{2}\left[\frac{T_3}{\alpha} - T_1\right]. \quad (3)$$

From (2) and (3) we obtain

$$\frac{A_{23}}{A_{12}} = \frac{3}{2} = \frac{T_3 - dT_1}{\alpha T_1 - T_1} = \frac{\alpha^2 - \alpha}{\alpha - 1} = \alpha\,;\ \alpha = \frac{3}{2}. \quad (4)$$

$$P_1V_1 = P_2V_2 \Rightarrow \frac{V_2}{V_1} = \frac{P_1}{P_2} = \alpha = \frac{3}{2}.$$

(1), (4)→(3): $$A_{23} = \frac{R\left(\frac{3}{2}+1\right)}{2}\left[\frac{3}{2}T_1 - T_1\right] = \frac{5}{8}RT_1 =$$
$$= \frac{5}{8}\cdot 8.31 \cdot 200 \approx 1039\ \text{J}.$$

4. Initial capacity $C = \frac{\varepsilon_0 S}{d_0}$. (1)

In the 2nd case, a capacitor with a plate is equivalent to two capaci-

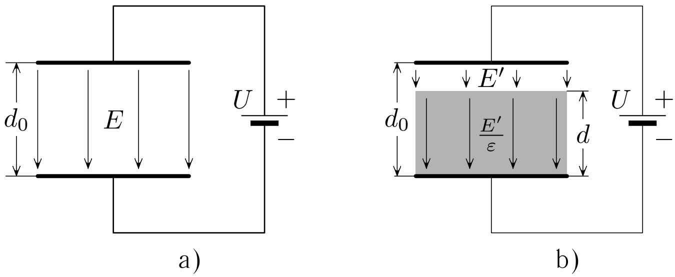

Fig. 27

tors connected in series (see fig. 27).

$$\frac{1}{C'} = \frac{1}{\varepsilon_0 \frac{S}{d_0/3}} + \frac{1}{\varepsilon\varepsilon_0 \frac{S}{2d_0/3}} \Rightarrow C' = \frac{3}{2+\varepsilon}\varepsilon\varepsilon_0\frac{S}{d_0}. \quad (2)$$

From (1) and (2), $C'/C = \frac{3\varepsilon}{2+\varepsilon} = 2 \Rightarrow 3\varepsilon = 4 + 2\varepsilon\,;\ \varepsilon = 4$. (3)

$$Ed_0 = E'\frac{d_0}{3} + \frac{E'}{\varepsilon}\cdot\frac{2}{3}d_0 = \frac{E'd_0}{3} + \frac{E'}{4}\cdot\frac{2}{3}d_0 \Rightarrow E' = 2E.$$

5. 1) Immediately after closing the switch, the current through the coil is zero (see fig. 28). $\Rightarrow I_1 = \mathscr{E}/R$.

2) In the steady state mode $I_0 = \mathscr{E}/R$. (1)

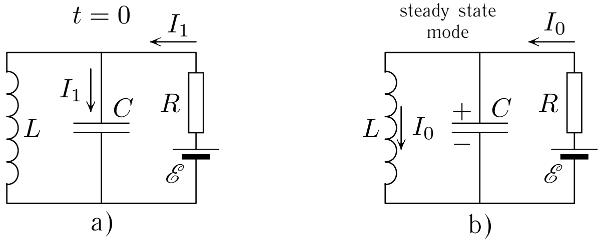

Fig. 28

3) Energy conservation law: $$\frac{LI_0^2}{2} = \frac{CU_C^2}{2} + \frac{L\left(\frac{3}{4}\,I_0\right)^2}{2}\,. \qquad (2)$$

(1)→(2): $$L\,\frac{\mathscr{E}^2}{R^2} = CU_C^2 + L\,\frac{9}{16}\,\frac{\mathscr{E}^2}{R^2}\,,\ U_C = \frac{\sqrt{7}\mathscr{E}}{4\cdot R}\cdot\left(\frac{L}{C}\right).$$

MATHEMATICS

Assessment Criteria
For the Final Stage of *Phystech.International* Olympiad

December, 2018.

Maximal total points for each problem are marked **bold**.

Problem Sets 1 and 2 for Pre-Graduation Class (9-th and 10-th Grades)

Problem	Assessment Criteria	Num pts
1.		**4**
2.		**5**
	The system of equations is composed	3
3.		**5**
	The system of inequalities is composed	2
4.		**5**
	A quadratic equation with respect to y is obtained and solved	3
5.		**5**
	A quadratic equation with respect to unknown length is obtained	3
6.		**5**
	The squares have been completed	2
7.		**8**
	The property of a bisector has been used	1
	The secants theorem has been applied (or the corresponding similarity of triangles)	1
	A trigonometric function of angle KLM (or of its half) is found	3

Problem Sets 1 and 2 for Graduation Class (11-th Grade)

Problem	Assessment Criteria	Num pts
1.		**5**
	The equation is factorized	2
	Both equations are solved correctly (without taking into account the domain of a function)	1
	The roots that belong to the domain of the function are chosen correctly	2
2.		**5**
	The characteristic properties of *both* progressions are used	2
3.		**5**
	A mistake in a trigonometric formula .... 0 points for the problem	
4.		**6**
	The condition for the lines to be perpendicular is written ($k_1k_2 = -1$)	1
	The condition for a line $y = kx + b$ to touch the parabola is written (i.e. the corresponding discriminant should be equal to 0) .... 2 points (for one or two equations)	
5.		**7**
	An algebraic equation with respect to *one* unknown trigonometric function that can be transformed into the quadratic equation is obtained	4
6.		**7**
	The domain of a function is determined	1
	The intervals for which the base of a logarithm is greater than 1 are determined	2
	The set is depicted	3
	The area of the set is found	1
7.		**7**
	The angles of a triangle PQR (at least two of them) are determined	3

MATHEMATICS

Problem Set #1, 9 and 10th Grade

1. ***Answer.*** 32 minutes.

Solution. Let v be the speed of a friend. Then speed of a former bus passenger is equal to $2.5v$, and the speed of the bus is equal to $6 \cdot 2.5v = 15v$. In 3 minutes that pass after the passenger has seen his friend the distance between them becomes equal to $3 \cdot (15v + v) = 48v$. When the passenger starts overtaking his friend, he reduces the distance between them by $1.5v$ every minute. So he needs $\frac{48v}{1.5v} = 32$ minutes for it.

2. ***Answer.*** 363.

Solution. Let the initial quantities of stamps in Jack's and Jill's possession be equal to x and y respectively. Then after the first exchange Jack has $\frac{1}{2}x + \frac{2}{11}y$ stamps and Jill has $\frac{1}{2}x + \frac{9}{11}y$ stamps. At the second exchange Jack gives away $\frac{1}{2}\left(\frac{x}{2} + \frac{2y}{11}\right)$ stamps and receives $\frac{2}{11}\left(\frac{1}{2}x + \frac{9}{11}y\right)$ stamps. It means that the amount of stamps in his possession becomes $\frac{15}{44}x + \frac{29}{121}y$. As the total amount of stamps does not change, Jill has $x + y - \left(\frac{15x}{44} + \frac{29y}{121}\right) = \frac{29}{44}x + \frac{92}{121}y$ stamps. It is known that $\frac{1}{2}x + \frac{2}{11}y = 110$ and $\frac{29}{44}x + \frac{92}{121}y = 334$. Solving this system of equations we get that $x = 88$ and $y = 363$. Therefore, the initial amount of stamps in Jill's possession is equal to 363.

3. ***Answer.*** $\frac{11}{113}$.

Solution. Let the numerator of the fraction be equal to k. Then its denominator is $k^2 - 8$. It is known that $\frac{k}{k^2-8} > \frac{1}{11}$ and $\frac{k+6}{k^2-7} < \frac{1}{5}$. As the denominator of a fraction is positive, and its numerator is integer, we deduce that $k \geqslant 3$. Therefore, we can multiply both parts of the first inequality by a *positive* expression $11\left(k^2 - 8\right)$ and multiply both parts of the second by a *positive* expression $5\left(k^2-7\right)$, thus getting a system

$$\begin{cases} k^2-11k-8<0, \\ k^2-5k-37>0. \end{cases} \Leftrightarrow \begin{cases} \dfrac{11-\sqrt{153}}{2}<k<\dfrac{11+\sqrt{153}}{2}, \\ k\in\left(-\infty;\dfrac{5-\sqrt{173}}{2}\right)\cup\left(\dfrac{5+\sqrt{173}}{2};+\infty\right). \end{cases}$$

Taking into account that k is an integer greater than 2, the first inequality yields $3\leqslant k\leqslant 11$, and from the second inequality it follows that $k\geqslant 10$. Combining the two conditions we get that either $k = 10$ or $k=11$. If $k=10$ the initial fraction is equal to $\frac{10}{92}$, so it is reducible by 2. If $k=11$ then the fraction equals $\frac{11}{113}$ which is irreducible. It is the only possibility that satisfies all the conditions given.

4. ***Answer.*** $(7;3)$, $\left(-189;\frac{1}{3}\right)$.

Solution. We start with factorizing left sides of both equations:

$$\begin{cases} x(y-1)\left(1+y+y^2\right)=182, \\ xy(y-1)=42. \end{cases}$$

As the right sides of the equations are different from zeroes (and left sides are equal to them) we can divide the first equation by the second one; we get $\frac{1+y+y^2}{y}=\frac{13}{3}$, $3y^2-10y+3=0$, therefore, $y=3$ or $y=\frac{1}{3}$. Substituting these values into the second equation of the initial system we can find the respective values of x:

- if $y=3$ then $9x-3x=42\Longleftrightarrow x=7$;
- if $y=\frac{1}{3}$ then $\frac{x}{9}-\frac{x}{3}=42\Longleftrightarrow x=-189$.

5. ***Answer.*** 30.

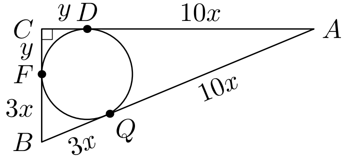

Fig. 1

Solution. Let us denote touch points of the incircle with sides AC and BC as D and F respectively (see fig. 1). Let $AQ=10x$, $CD = y$. As $AQ:QB=10:3$, we get that $BQ=3x$. We also get $AD=AQ=10x$, $CF=CD=y$, $BF=BQ=3x$ since the segments of tangent lines drawn to a circle from one point are equal to each other. As perimeter of the triangle is 30, we get that $2y+26x=$

$= 30$, $y = 15 - 13x$. So, $AC = 10x + y = 15 - 3x$, $BC = 3x + y = 15 - 10x$, $AB = 3x + 10x = 13x$. Then Pythagorean theorem yields $(15 - 3x)^2 + (15 - 10x)^2 = (13x)^2 \iff 2x^2 + 13x - 15 = 0$, and so $x = 1$ or $x = -\frac{15}{2}$. The negative value of x is not suitable. Thus, $x = 1$, $AC = 12$, $BC = 5$, $S_{\triangle ABC} = \frac{1}{2} AC \cdot BC = 30$.

6. ***Answer.*** $(15; -6)$.

Solution. Making out the exact squares we get

$$\begin{cases} (x - 12)^2 + (y + 8)^2 < 15, \\ (x - 19)^2 + (y + 4)^2 < 23. \end{cases}$$

As squares are non-negative we can conclude that $(x - 12)^2 < 15$ and $(x - 19)^2 < 23$. The only integer value of x that satisfies these inequalities is $x = 15$. Substituting it into the system we get $\begin{cases} (y + 8)^2 < 6, \\ (y + 4)^2 < 7 \end{cases}$. The only value of y that suits here is $y = -6$. So we get the only pair of integers $(15; -6)$ that satisfy the given inequalities.

7. ***Answer.*** 8.

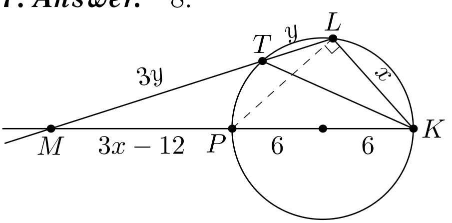

Fig. 2

Solution. Let $LT = y$. Then $MT = 3y$ (see fig. 2). As KT is a bisector of a triangle, $KL : KM = LT : MT = y : 3y = 1 : 3$. So, if $KL = x$, then $KM = 3x$. ML and MK are two secant lines to a circle drawn from one point; therefore, $MT \cdot ML = MK \cdot MP$ (P is the intersection point of KM with the circle), i.e. $3y \cdot 4y = 3x \cdot (3x - 12) \iff 4y^2 = 3x^2 - 12x$. As KP is a diameter, it subtends a right angle at point L, and so $\cos \angle LKP = \frac{KL}{KP} = \frac{x}{12}$. Then cosine theorem for triangle KLM yields $(4y)^2 = x^2 + (3x)^2 - 2 \cdot x \cdot 3x \cdot \frac{x}{12}$.

Substituting here the expression for $4y^2$ obtained above we get $12x^2 - 48x = 10x^2 - \frac{1}{2} x^3 \iff x(x - 8)(x + 12) = 0$. This equation has the only positive root, that is $x = 8$. Consequently, $KL = x = 8$.

Problem Set #2, 9 and 10th Grade

1. ***Answer.*** 39 minutes.

Solution. Let v be the speed of a friend. Then speed of a former bus passenger is equal to $1.8v$, and the speed of the bus is equal to $11 \cdot 1.8v = 19.8v$. In 1.5 minutes that pass after the passenger has seen his friend the distance between them becomes equal to $1.5 \cdot (19.8v + v) = 31.2v$. When the passenger starts overtaking his friend, he reduces the distance between them by $0.8v$ every minute. So he needs $\dfrac{31.2v}{0.8v} = 39$ minutes for it.

2. ***Answer.*** 147.

Solution. Let the initial quantities of stamps in Jack's and Jill's possession be equal to x and y respectively. Then after the first exchange Jack has $\frac{2}{3}x + \frac{3}{7}y$ stamps and Jill has $\frac{1}{3}x + \frac{4}{7}y$ stamps.

At the second exchange Jack gives away $\frac{1}{3}\left(\frac{2}{3}x + \frac{3}{7}y\right)$ stamps and receives $\frac{3}{7}\left(\frac{1}{3}x + \frac{4}{7}y\right)$ stamps. It means that the amount of stamps in his possession becomes $\frac{37}{63}x + \frac{26}{49}y$. As the total amount of stamps does not change, Jill has $x + y - \left(\frac{37}{63}x + \frac{26}{49}y\right) = \frac{26}{63}x + {}$ $+ \frac{23}{49}y$ stamps. It is known that $\frac{2}{3}x + \frac{3}{7}y = 273$ and $\frac{26}{63}x + \frac{23}{49}y =$ $= 199$. Solving this system of equations we get that $x = 315$ and $y = 147$. Therefore, the initial amount of stamps in Jill's possession is equal to 147.

3. ***Answer.*** $\dfrac{8}{61}$.

Solution. Let the numerator of the fraction be equal to k. Then its denominator is $k^2 - 3$. It is known that $\dfrac{k}{k^2-3} > \dfrac{1}{9}$ and $\dfrac{k+3}{k^2-2} < \dfrac{1}{5}$. As the denominator of a fraction is positive, and its numerator is integer, we deduce that $k \geqslant 2$. Therefore, we can multiply both parts of the first inequality by a *positive* expression $9\left(k^2-3\right)$ and multiply both parts of the second by a *positive* expression $5\left(k^2-2\right)$, thus getting a system

$$\begin{cases} k^2 - 9k - 3 < 0, \\ k^2 - 5k - 17 > 0. \end{cases} \Longleftrightarrow \begin{cases} \dfrac{9-\sqrt{93}}{2} < k < \dfrac{9+\sqrt{93}}{2}, \\ k \in \left(-\infty; \dfrac{5-\sqrt{93}}{2}\right) \cup \left(\dfrac{5+\sqrt{93}}{2}; +\infty\right). \end{cases}$$

Taking into account that k is an integer greater than 1, the first inequality yields $2 \leqslant k \leqslant 9$, and from the second inequality it follows that $k \geqslant 8$. Combining the two conditions we get that either $k = 8$ or $k = 9$. If $k = 9$ the initial fraction is equal to $\frac{9}{78}$, so it is reducible by 3. If $k = 8$ then the fraction equals $\frac{8}{61}$ which is irreducible. It is the only possibility that satisfies all the conditions given.

4. ***Answer.*** $(10; -2)$, $\left(-80; -\frac{1}{2}\right)$.

Solution. We start with factorizing left sides of both equations:

$$\begin{cases} x(1+y)\left(1-y+y^2\right) = -70, \\ xy(1+y) = 20. \end{cases}$$

As the right sides of the equations are different from zeroes (and left sides are equal to them) we can divide the first equation by the second one; we get $\frac{1-y+y^2}{y} = -\frac{7}{2}$, $2y^2 + 5y + 2 = 0$, therefore, $y = -2$ or $y = -\frac{1}{2}$. Substituting these values into the first equation of the initial system we can find the respective values of x:

- if $y = -2$ then $x - 8x = -70 \iff x = 10$;
- if $y = -\frac{1}{2}$ then $x - \frac{1}{8}x = -70 \iff x = -80$.

5. ***Answer.*** 60.

Solution. Let us denote touch points of the incircle with sides AC and BC as D and F respectively. Let $AQ = 5x$, $CD = y$ (see fig. 3). As $AQ : QB = 5 : 12$, we get that $BQ = 12x$. We also get $AD = AQ = 5x$, $CF = CD = y$, $BF = BQ = 12x$ since the segments of tangent lines drawn to a circle from one point are equal to each other. As perimeter of the triangle is 40, we get that $2y + 34x = 40$, $y = 20 - 17x$. So, $AC = 5x + y = 20 - 12x$, $BC = 12x + y = 20 - 5x$, $AB = 5x + 12x = 17x$. Then Pythagorean theorem yields $(20-5x)^2 + (20-12x)^2 = (17x)^2 \iff 3x^2 + 17x - 20 = 0$, and so $x = 1$ or $x = -\frac{20}{3}$. The negative value of x is not suitable. Thus, $x = 1$, $AC = 8$, $BC = 15$, $S_{\triangle ABC} = \frac{1}{2} AC \cdot BC = 60$.

6. ***Answer.*** $(5; -11)$.

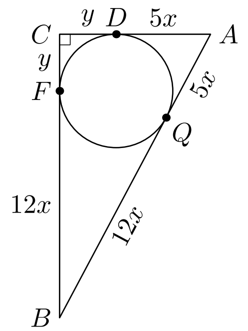

Fig. 3

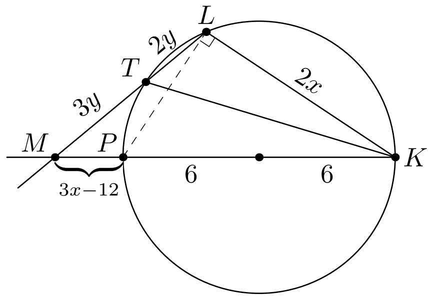

Fig. 4

Solution. Making out the exact squares we get

$$\begin{cases} (x-2)^2 + (y+13)^2 < 14, \\ (x-9)^2 + (y+9)^2 < 22. \end{cases}$$

As squares are non-negative we can conclude that $(x-2)^2 < 14$ and $(x-9)^2 < 22$. The only integer value of x that satisfies these inequalities is $x = 5$. Substituting it into the system we get $\begin{cases} (y+13)^2 < 5, \\ (y+9)^2 < 6 \end{cases}$. The only value of y that suits here is $y = -11$. So we get the only pair of integers $(5; -11)$ that satisfy the given inequalities.

7. ***Answer.*** 10.

Solution. Let $LT = 2y$. Then $MT = 3y$ (see fig. 4). As KT is a bisector of a triangle, $KL : KM = LT : MT = 2y : 3y = 2 : 3$. Let $KL = 2x$, then $KM = 3x$. ML and MK are two secant lines to a circle drawn from one point; therefore, $MT \cdot ML = MK \cdot MP$ (P is the intersection point of KM with the circle), i.e. $3y \cdot 5y = 3x \cdot (3x - 12) \iff 5y^2 = 3x^2 - 12x$. As KP is a diameter, it subtends a right angle at point L, and so $\cos\angle LKP = \frac{KL}{KP} = \frac{x}{6}$. Then cosine theorem for triangle KLM yields $(5y)^2 = (2x)^2 + (3x)^2 - 2 \cdot 2x \cdot 3x \cdot \frac{x}{6}$. Substituting here the expression for $5y^2$ obtained above we get $15x^2 - 60x = 13x^2 - 2x^3 \iff x(x -$

$-5)(x+6)=0$. This equation has the only positive root, that is $x=5$. Consequently, $KL=2x=10$.

Problem Set #1, 11th Grade

1. ***Answer.*** $7;\ \sqrt{2}$.

Solution. Moving all terms to the left side and factorizing, we get $\left(x^4-4\right)\left(2^{11-x}-2^{\sqrt{2x+2}}\right)=0$. The first factor is equal to 0 for $x=\pm\sqrt{2}$, and only $x=\sqrt{2}$ belongs to the domain of the second factor. The second factor equals 0 if

$$\sqrt{2+2x}=11-x \Longleftrightarrow \begin{cases} 2+2x=x^2-22x+121, \\ 11-x\geqslant 0 \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \left[\begin{array}{l} x=17, \\ x=7, \end{array}\right. \\ x\leqslant 11 \end{cases} \Longleftrightarrow x=7.$$

Finally, we get $x=7$ and $x=\sqrt{2}$.

2. ***Answer.*** -4 or 1.

Solution. As $c-a$, $2a-b$ and $a+b$ form an arithmetic progression, its middle term is equal to half-sum of two other terms; therefore, $2(2a-b)=(c-a)+(a+b) \Longleftrightarrow c=4a-3b$. For the sequence to be a geometric progression, any of its terms squared should be equal to the product of the two neighbors. So, $b^2=ac$, or, taking into account that $c=4a-3b$, we get $a(4a-3b)=b^2$, $b^2+3ab-4a^2=0$. Solving this equation as quadratic equation with respect to b, we obtain that either $b=-4a$ or $b=a$. The common ratio q of a geometric progression is equal to $\frac{b}{a}$, and so we get that either $q=-4$ or $q=1$.

3. ***Answer.*** $\frac{\sqrt{3}}{8}$.

Solution. Transforming product of trigonometric functions into their sum we get

$$\cos 10^\circ \cos 50^\circ=\frac{1}{2}\cos 40^\circ+\frac{1}{2}\cos 60^\circ=\frac{1}{2}\cos 40^\circ+\frac{1}{4}\,.$$

Therefore, the initial expression is equal to

$$\left(\frac{1}{2}\cos 40^\circ + \frac{1}{4}\right)\cos 70^\circ = \frac{1}{2}\cos 40^\circ \cos 70^\circ + \frac{1}{4}\cos 70^\circ = \\ = \frac{1}{4}\cos 110^\circ + \frac{1}{4}\cos 30^\circ + \frac{1}{4}\cos 70^\circ.$$

As $\cos 110^\circ = \cos(180^\circ - 70^\circ) = -\cos 70^\circ$, we finally get $\frac{1}{4}\cos 30^\circ = \frac{\sqrt{3}}{8}$.

4. ***Answer.*** $\left(0; \frac{55}{6}\right)$.

Solution. Let us choose a point with ordinate a on y-axis. It is obvious that the lines we are considering are not parallel to coordinate axes. Then product of slopes of these lines equals (-1). Let us denote these slopes as k and $-\frac{1}{k}$ respectively. Consequently, the equations of the lines are $y = kx + a$ and $y = -\frac{1}{k}x + a$. Each of them has exactly one common point with the parabola, therefore systems

$$\begin{cases} y = 7 - 5x - 3x^2, \\ y = kx + a \end{cases} \quad \text{and} \quad \begin{cases} y = 7 - 5x - 3x^2, \\ y = -\frac{1}{k}x + a \end{cases}$$

have one solution each. Equating the right sides in each of the systems, we get that equations

$$kx + a = 7 - 5x - 3x^2 \quad \text{and} \quad -\frac{1}{k}x + a = 7 - 5x - 3x^2$$

must have exactly one solution each.

Let us consider the first equation separately. It is equivalent to $3x^2 + (k+5)x + (a-7) = 0$. For it to have one solution, its discriminant has to be 0, and so $(k+5)^2 - 12(a-7) = 0$. In the same way the second equation yields $\left(\frac{1}{k} - 5\right)^2 - 12(a-7) = 0$.

As k and a must satisfy both of the obtained equations, we can subtract the second of them from the first; it yields $(k+5)^2 - \left(\frac{1}{k} - 5\right)^2 = 0$; hence $k + 5 = -\frac{1}{k} + 5$ or $k + 5 = \frac{1}{k} - 5$. The former equation yields $k^2 = -1$, and so it has no solutions. The latter equation is equivalent to $k^2+10k-1=0$, thus $k=-5\pm\pm\sqrt{26}$. Substituting these values of k into $(k+5)^2=12(a-7)$ we get $12(a-7)=26$, $a=\dfrac{55}{6}$.

5. ***Answer.*** $7\sqrt{23}$.

Solution. Let $\angle KLM=\varphi$, then $\angle LMN=180^\circ-\varphi$ (see fig. 5). Cosine theorem applied to triangles KLP and MNP yields $KP^2==81+4-36\cos\varphi$, $NP^2=81+4+36\cos\varphi$. Then we use cosine theorem for triangle KPN:

$$16=KN^2=(85+36\cos\varphi)+(85-36\cos\varphi)-$$
$$-2\sqrt{(85+36\cos\varphi)(85-36\cos\varphi)}\cdot\frac{11}{12}\iff$$
$$\iff\sqrt{85^2-36^2\cos^2\varphi}=84\iff$$
$$\iff 36^2\cos^2\varphi=85^2-84^2=(85-84)(85+84)\iff\cos^2\varphi=\frac{13^2}{36^2}.$$

Hence $\sin^2\varphi=\left(1-\dfrac{13}{36}\right)\left(1+\dfrac{13}{36}\right)=\dfrac{49\cdot 23}{36^2}$, $\sin\varphi=\dfrac{7\sqrt{23}}{36}$, and so the area of the parallelogram is equal to $KL\cdot LM\cdot\sin\varphi=7\sqrt{23}$.

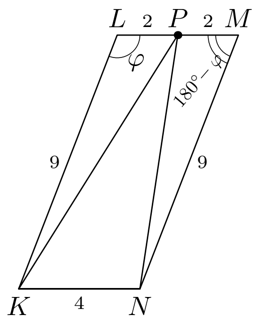

Fig. 5

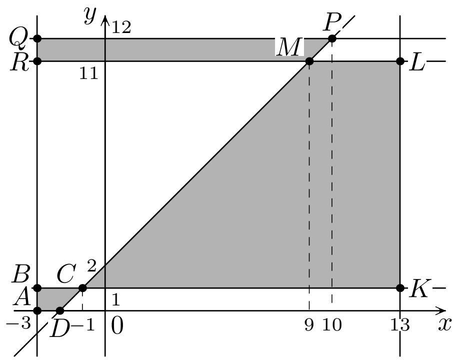

Fig. 6

6. ***Answer.*** The set is depicted on fig. 6; its area is equal to 104.

Solution. Let us consider the two options available.

a) The base of the logarithm is greater than 1. It happens if and only if

$$|y-2|-2|y-4|+6>1 \iff |2y-8|<|y-2|+5 \iff$$
$$\iff \begin{cases} 2y-8<|y-2|+5, \\ 2y-8>-|y-2|-5 \end{cases} \iff \begin{cases} |y-2|>2y-13, \\ |y-2|>3-2y \end{cases} \iff$$
$$\iff \left\{ \begin{array}{l} \left[\begin{array}{l} y-2>2y-13, \\ y-2<13-2y, \end{array} \right. \\ \left[\begin{array}{l} y-2>3-2y, \\ y-2<2y-3 \end{array} \right. \end{array} \right. \iff \left\{ \begin{array}{l} \left[\begin{array}{l} y<11, \\ y<5, \end{array} \right. \\ \left[\begin{array}{l} y>\frac{5}{3}, \\ y>1 \end{array} \right. \end{array} \right. \iff$$
$$\iff \begin{cases} y<11, \\ y>1 \end{cases} \iff 1<y<11.$$

For these values of y the initial inequality yields $x+3>1+y$, i.e. $y<x+2$. If we also take into account that $x<13$, we obtain a trapezoid $CKLM$, the coordinates of its vertices being $C(-1;1)$, $K(13;1)$, $L(13;11)$, $M(9;11)$ (see fig. 6).

b) The base of the logarithm is between 0 and 1. To determine the values of y for which it happens we solve the inequality $|y-2|-2|y-4|+6>0$ and then exclude interval $[1;11]$ from its solution set.

$$|y-2|-2|y-4|+6>0 \iff |2y-8|<|y-2|+6 \iff$$
$$\iff \begin{cases} 2y-8<|y-2|+6, \\ 2y-8>-|y-2|-6 \end{cases} \iff \begin{cases} |y-2|>2y-14, \\ |y-2|>2-2y \end{cases} \iff$$
$$\iff \left\{ \begin{array}{l} \left[\begin{array}{l} y-2>2y-14, \\ y-2<14-2y, \end{array} \right. \\ \left[\begin{array}{l} y-2>2-2y, \\ y-2<2y-2 \end{array} \right. \end{array} \right. \iff \left\{ \begin{array}{l} \left[\begin{array}{l} y<12, \\ y<\frac{16}{3}, \end{array} \right. \\ \left[\begin{array}{l} y>\frac{4}{3}, \\ y>0 \end{array} \right. \end{array} \right. \iff$$
$$\iff \begin{cases} y<12, \\ y>0 \end{cases} \iff 0<y<12.$$

Thus the base of the logarithm is between 0 and 1 for $y \in (0;1) \cup \cup (11;12)$. For these values of y we conclude that $x+3<1+$

$+\,y$, i.e. $y > x + 2$. If we add that $x > -3$ (that follows from the domain of the initial inequality), we obtain two trapezoids $ABCD$ and $MPQR$, their vertices situated in points $A(-3;0)$, $B(-3;1)$, $C(-1;1)$, $D(-2;0)$, $M(9;11)$, $P(10;12)$, $Q(-3;12)$, $R(-3;11)$. You can see the set on fig. 6 (all the boundaries do not belong to the set).

The area of this set is equal to sum of areas of all the trapezoids. $S_{ABCD} = \dfrac{1+2}{2} \cdot 1 = 1.5$, $S_{CKLM} = \dfrac{14+4}{2} \cdot 10 = 90$, $S_{ABCD} = = \dfrac{12+13}{2} \cdot 1 = 12.5$; hence $S_{\text{total}} = 104$.

7. ***Answer.*** 4.

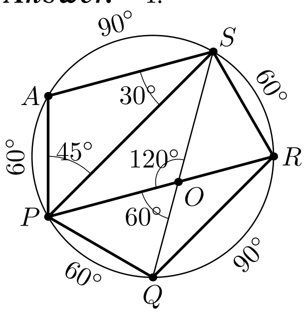

Fig. 7

Solution. Angle QOP is $60°$ and is the angle between the chords that intersect; therefore, it is equal to one half of the sum of arcs PQ and RS (see fig. 7). As these arcs are equal to each other (they are arcs between parallel chords QR and PS), we conclude that each of them is equal to $60°$. Chords AS and PR are also parallel to each other (as they are bases of a trapezoid), and so arc AP is as well equal to $60°$. Chords PR and PS are equal to each other; and so are the corresponding arcs, that is, arc PQR equals arc PAS. From here follows that $\overset{\smile}{QR} = \overset{\smile}{AS}$. As the whole circle constitutes $360°$, we get that $\overset{\smile}{QR} = \overset{\smile}{AS} = 90°$.

Angle APS is inscribed into circle Ω, and so it is equal to $\frac{1}{2}\,\overset{\smile}{AS} = = \frac{1}{2} \cdot 90° = 45°$. In the same way $\angle ASP = \frac{1}{2}\,\overset{\smile}{AP} = 30°$. Then $\angle PAS = 180° - \angle APS - \angle ASP = 105°$. Let the diameter of the circle be equal to d. Then sine theorem used for triangle APS yields $AP = d \sin 30° = \dfrac{d}{2}$, $AS = d \sin 45° = \dfrac{d}{\sqrt{2}}$. For expressing the area of triangle APS we also need that $\sin 105° = \sin(45° + 60°) = = \sin 45° \cos 60° + \cos 45° \sin 60° = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$.

Hence the area of triangle APS is equal to $\frac{1}{2}\,AP \cdot AS \cdot \sin 105° =$

$= \dfrac{d^2\left(\sqrt{3}+1\right)}{16}$. Equating it to $4\left(\sqrt{3}+1\right)$ yields $d^2 = 64$, so $d = 8$ and the radius is equal to 4.

Problem Set #2, 11th Grade

1. ***Answer.*** -5; $-\sqrt{3}$.

Solution. Moving all terms to the left side and factorizing, we get $\left(x^4-9\right)\left(3^{\sqrt{1-3x}}-3^{x+9}\right)=0$. The first factor is equal to 0 for $x = \pm\sqrt{3}$, and only $x = -\sqrt{3}$ belongs to the domain of the second factor. The second factor equals 0 if

$$\sqrt{1-3x} = x+9 \iff \begin{cases} 1-3x = x^2+18x+81, \\ x+9 \geqslant 0 \end{cases} \iff$$

$$\iff \begin{cases} \left[\begin{array}{l} x = -16, \\ x = -5, \end{array}\right. \\ x \geqslant -9 \end{cases} \iff x = -5.$$

Finally we get $x = -5$ and $x = -\sqrt{3}$.

2. ***Answer.*** -3 or 1.

Solution. As $3c-2a$, $a-b$ and $a-2c$ form an arithmetic progression, its middle term is equal to half-sum of two other terms; therefore, $2(a-b) = (a-2c)+(3c-2a) \iff c = 3a-2b$. For the sequence to be a geometric progression, any of its terms squared should be equal to the product of the two neighbors. So, $b^2 = ac$, or, taking into account that $c = 3a-2b$, we get $a(3a-2b) = b^2$, $b^2+2ab-3a^2 = 0$. Solving this equation as quadratic equation with respect to b, we obtain that either $b = -3a$ or $b = a$. The common ratio q of a geometric progression is equal to $\dfrac{b}{a}$, and so we get that either $q = -3$ or $q = 1$.

3. ***Answer.*** $\dfrac{1}{8}$.

Solution. Transforming product of trigonometric functions into their sum we get

$$\sin 10^\circ \sin 50^\circ = \frac{1}{2}\cos 40^\circ - \frac{1}{2}\cos 60^\circ = \frac{1}{2}\cos 40^\circ - \frac{1}{4}\,.$$

Therefore, the initial expression is equal to

$$\left(\frac{1}{2}\cos 40^\circ - \frac{1}{4}\right)\sin 70^\circ = \frac{1}{2}\cos 40^\circ \sin 70^\circ - \frac{1}{4}\sin 70^\circ =$$
$$= \frac{1}{4}\sin 110^\circ + \frac{1}{4}\sin 30^\circ - \frac{1}{4}\sin 70^\circ.$$

As $\sin 110^\circ = \sin(180^\circ - 70^\circ) = \sin 70^\circ$, we finally get

$$\frac{1}{4}\sin 30^\circ = \frac{1}{8}\,.$$

4. ***Answer.*** $\left(0; \frac{53}{16}\right)$.

Solution. Let us choose a point with ordinate a on y-axis. It is obvious that the lines we are considering are not parallel to coordinate axes. Then product of slopes of these lines equals (-1). Let us denote these slopes as k and $-\frac{1}{k}$ respectively. Consequently, the equations of the lines are $y = kx + a$ and $y = -\frac{1}{k}x + a$. Each of them has exactly one common point with the parabola, therefore systems

$$\begin{cases} y = 1 + 6x - 4x^2, \\ y = kx + a \end{cases} \quad \text{and} \quad \begin{cases} y = 1 + 6x - 4x^2, \\ y = -\frac{1}{k}x + a \end{cases}$$

have one solution each. Equating the right sides in each of the systems, we get that equations

$$kx + a = 1 + 6x - 4x^2 \quad \text{and} \quad -\frac{1}{k}x + a = 1 + 6x - 4x^2$$

must have exactly one solution each.

Let us consider the first equation separately. It is equivalent to $4x^2 + (k-6)x + (a-1) = 0$. For it to have one solution, its discriminant has to be 0, and so $(k-6)^2 - 16(a-1) = 0$. In the same way the second equation yields $\left(\frac{1}{k} + 6\right)^2 - 16(a-1) = 0$.

As k and a must satisfy both of the obtained equations, we can subtract the second of them from the first, thus getting $(k-6)^2 - \left(\frac{1}{k} + 6\right)^2 = 0$; hence $k - 6 = -\frac{1}{k} - 6$ or $k - 6 = \frac{1}{k} + 6$. The former equation yields $k^2 = -1$, and so it has no solutions. The latter equation is equivalent to $k^2 - 12k - 1 = 0$, thus $k = 6 \pm \sqrt{37}$. Substituting these values of k into $(k-6)^2 = 16(a-1)$ we get $16(a-1) = 37$, $a = \frac{53}{16}$.

5. ***Answer.*** $2\sqrt{17}$.

Solution. Let $\angle KLM = \varphi$, then $\angle LMN = 180° - \varphi$ (see fig. 8). Cosine theorem applied to triangles KLP and MNP yields $KP^2 = 25 + 9 - 30\cos\varphi$, $NP^2 = 25 + 9 + 30\cos\varphi$. Then we use cosine theorem for triangle KPN:

$$36 = KN^2 = (34 + 30\cos\varphi) + (34 - 30\cos\varphi) - 2\sqrt{(34 + 30\cos\varphi)(34 - 30\cos\varphi)} \cdot \frac{8}{9} \iff$$

$$\iff \sqrt{34^2 - 30^2\cos^2\varphi} = 18 \iff$$

$$\iff 30^2\cos^2\varphi = 34^2 - 18^2 = (34-18)(34+18) \iff$$

$$\iff \cos^2\varphi = \frac{13 \cdot 16}{15^2}.$$

Hence $\sin^2\varphi = 1 - \frac{13 \cdot 16}{15^2} = \frac{17}{15^2}$, $\sin\varphi = \frac{\sqrt{17}}{15}$, and so the area of the parallelogram is equal to $KL \cdot LM \cdot \sin\varphi = 2\sqrt{17}$.

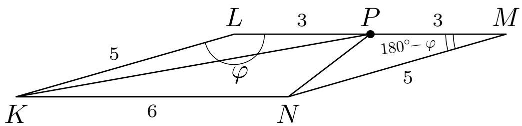

Fig. 8

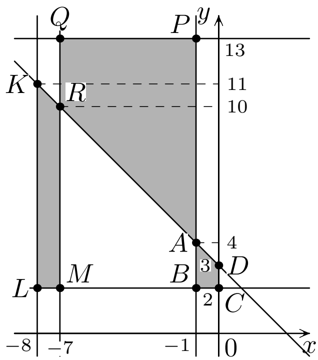

Fig. 9

6. ***Answer.*** The set is depicted on fig. 9; its area is equal to 46.

Solution. Let us consider the two options available.

a) The base of the logarithm is greater than 1. It happens if and only if

$$|x+2|-2|x+3|+4>1 \iff |2x+6|<|x+2|+3 \iff$$
$$\iff \begin{cases} 2x+6<|x+2|+3, \\ 2x+6>-|x+2|-3 \end{cases} \iff \begin{cases} |x+2|>2x+3, \\ |x+2|>-2x-9 \end{cases} \iff$$
$$\iff \begin{cases} \left[\begin{array}{l} x+2>2x+3, \\ x+2<-2x-3, \end{array}\right. \\ \left[\begin{array}{l} x+2>-2x-9, \\ x+2<2x+9 \end{array}\right. \end{cases} \iff \begin{cases} \left[\begin{array}{l} x<-1, \\ x<-\dfrac{5}{3}, \end{array}\right. \\ \left[\begin{array}{l} x>-\dfrac{11}{3}, \\ x>-7 \end{array}\right. \end{cases} \iff$$
$$\iff \begin{cases} x<-1, \\ x>-7 \end{cases} \iff -7<x<-1.$$

We get that $y-2>1-x$, i.e. $y>3-x$. If we also take into account that $y<13$, we obtain a trapezoid $APQR$, the coordinates of its vertices being $A(-1;4)$, $P(-1;13)$, $Q(-7;13)$, $R(-7;10)$.

b) The base of the logarithm is between 0 and 1. To determine the values of x for which it happens we solve the inequality $|x+2|-2|x+3|+4>0$ and then exclude an interval $[-7;-1]$ from its solution set.

$$|x+2|-2|x+3|+4>0 \iff |2x+6|<|x+2|+4 \iff$$
$$\iff \begin{cases} 2x+6<|x+2|+4, \\ 2x+6>-|x+2|-4 \end{cases} \iff \begin{cases} |x+2|>2x+2, \\ |x+2|>-2x-10 \end{cases} \iff$$
$$\iff \begin{cases} \left[\begin{array}{l} x+2>2x+2, \\ x+2<-2x-2, \end{array}\right. \\ \left[\begin{array}{l} x+2>-2x-10, \\ x+2<2x+10 \end{array}\right. \end{cases} \iff \begin{cases} \left[\begin{array}{l} x<0, \\ x<-\dfrac{4}{3}, \end{array}\right. \\ \left[\begin{array}{l} x>-4, \\ x>-8 \end{array}\right. \end{cases} \iff$$
$$\iff \begin{cases} x<0, \\ x>-8 \end{cases} \iff -8<x<0.$$

Thus the base of the logarithm is between 0 and 1 for $x \in (-8;-7) \cup (-1;0)$. For these values of x we conclude that $y-2<1-x$, i.e. $y<3-x$. If we add that $y>$

> 2 (that follows from the domain of the initial inequality), we obtain two trapezoids $ABCD$ and $KLMR$, their vertices situated in points $A(-1;4)$, $B(-1;2)$, $C(0;2)$, $D(0;3)$, $R(-7;10)$, $K(-8;11)$, $L(-8;2)$, $M(-7;2)$.

The area of this set is equal to sum of areas of all the trapezoids. $S_{ABCD} = \dfrac{1+2}{2} \cdot 1 = 1.5$, $S_{APQR} = \dfrac{9+3}{2} \cdot 6 = 36$, $S_{KLMR} = \dfrac{9+8}{2} \cdot 1 = 8.5$; hence $S_{\text{total}} = 46$.

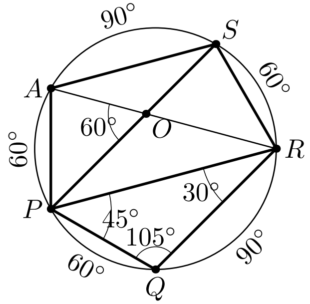

Fig. 10

7. ***Answer.*** 6.

Solution. Angle AOP is the angle between the chords that intersect; therefore, it is equal to one half of the sum of arcs AP and RS. As these arcs are equal to each other (they are arcs between parallel chords PR and AS), we conclude that each of them is equal to $60°$. Chords QR and PS are also parallel to each other (as they are bases of a trapezoid), and so arc PQ is equal to $60°$ as well. Chords PR and PS are equal to each other; and so are the corresponding arcs, that is, arc PQR equals arc PAS. From here follows that $\overset{\smile}{QR} = \overset{\smile}{AS}$. As the whole circle is $360°$, we get that $\overset{\smile}{QR} = \overset{\smile}{AS} = 90°$.

Angle QPR is inscribed into circle Ω, and so it is equal to $\frac{1}{2}\overset{\smile}{QR} = \frac{1}{2} \cdot 90° = 45°$. In the same way $\angle PRQ = \frac{1}{2}\overset{\smile}{PQ} = 30°$. Then $\angle PQR = 180° - \angle QPR - \angle PRQ = 105°$. Let the diameter of the circle be equal to d. Then sine theorem used for triangle PQR yields $PQ = d\sin 30° = \frac{d}{2}$, $QR = d \sin 45° = \frac{d}{\sqrt{2}}$. For expressing the area of triangle PQR we also need that $\sin 105° = \sin(45° + 60°) = \sin 45° \cos 60° + \cos 45° \sin 60° = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$.

Hence the area of triangle PQR is equal to $\frac{1}{2}PQ \cdot QR \cdot \sin 105° = \dfrac{d^2\left(\sqrt{3}+1\right)}{16}$. Equating it to $9\left(\sqrt{3}+1\right)$ yields $d^2 = 144$, so $d = 12$ and the radius is equal to 6.

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